NONSCATTERING SOLUTIONS TO THE L^2-SUPERCRITICAL NLS EQUATIONS

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ABSTRACT. We investigate the nonlinear Schrödinger equation $iu_t + \Delta u + |u|^{p-1}u = 0$ with $1+\frac{4}{N} < p < 1+\frac{4}{N-2}$ (when $N = 1, 2$, $1+\frac{4}{N} < p < \infty$) in energy space H^1 and study the divergent property of infinite-variance and nonradial solutions. If $M(u)^{\frac{1-s_c}{s_c}}E(u) < M(Q)^{\frac{1-s_c}{s_c}}E(Q)$ and $\|u_0\|_2^{\frac{1-s_c}{s_c}}\|\nabla u_0\|_2 > \|Q\|_2^{\frac{1-s_c}{s_c}}\|\nabla Q\|_2$, then either $u(t)$ blows up in finite forward time, or $u(t)$ exists globally for positive time and there exists a time sequence $t_n \to +\infty$ such that $\|\nabla u(t_n)\|_2 \to +\infty$. Here Q is the ground state solution of $-Q+\Delta Q+|Q|^{p-1}Q = 0$. A similar result holds for negative time. This extend the result of the 3D cubic Schrödinger equation in [7] to the general mass-supercritical and energy-subcritical case .

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Keywords: Nonlinear Schrödinger equation; Blow-up solution; Infinite variance; Mass-supercritical; Energy-subcritical

1. INTRODUCTION

We consider the following Cauchy problem of a nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u + |u|^{p-1}u = 0, \quad (x,t) \in R^N \times R, \\ u(x,0) = u_0(x) \in H^1(R^N). \end{cases} \tag{1.1}$$

It is well known from Ginibre and Velo [3] that, equation (1.1) is locally well-posed in H^1. That is for $u_0 \in H^1$, there exist $0 < T \le \infty$ and a unique solution $u(t) \in C([0,T); H^1)$ to (1.1). When $T = \infty$, we say that the solution is positively global; while on the other hand, we have $\lim_{t\uparrow T}\|\nabla u(t)\|_2 \to \infty$ and call that this solution blows up in finite positive time. Solutions of (1.1) admits the following conservation laws in energy space H^1 :

$$L^2 - norm: \quad M(u)(t) \equiv \int |u(x,t)|^2dx = M(u_0);$$

$$Energy: \quad E(u)(t) \equiv \frac{1}{2}\int |\nabla u(x,t)|^2dx - \frac{1}{p+1}\int |u(x,t)|^{p+1}dx = E(u_0);$$

$$Momentum: \quad P(u)(t) \equiv Im\int \overline{u}(x,t)\nabla u(x,t)dx = P(u_0).$$

Note that equation (1.1) is invariant under the scaling $u(x,t) \to \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t)$ which also leaves the homogeneous Sobolev norm $\dot{H}^{s_c}$ invariant with $s_c = \frac{N}{2} - \frac{2}{p-1}$. It is classical from the conservation of the energy and the L^2 norm that for $s_c < 0$, the equation is subcritical and all H^1 solutions are global and H^1 bounded. The smallest power for which blow up may occur is $p = 1 + \frac{4}{N}$ which is referred to as the L^2 critical case corresponding to $s_c = 0$ [4] [12]. The case $0 < s_c < 1$ is called the L^2 supercritical and H^1 subcritical or the Mass-supercritical and Energy-subcritical case. In fact, we are concerning in this paper with the case $0 < s_c < 1$.

For the 3D cubic nonlinear Schrödinger equation with $s_c = \frac{1}{2}$ and $p = 3$, there have been several results on either scattering or blow-up solutions. In Holmer and Roudenko [6], the authors proved that if $u_0 \in H^1$ is radial, $M(u)E(u) < M(Q)E(Q)$ and $\|\nabla u_0\|_2 \|u_0\|_2 < \|\nabla Q\|_2 \|Q\|_2$, then the solution $u(t)$ is globally well-posed and scattering; They further showed that if $M(u)E(u) < M(Q)E(Q)$ and $\|\nabla u_0\|_2 \|u_0\|_2 > \|\nabla Q\|_2 \|Q\|_2$, then the solution blows up in finite time, provided that either the initial data has finite variance or is radial. The radial case is an extension of a result of Ogawa and Tsutsumi [14] who proved the case $E(u) < 0$. Then in [2], also for the 3D cubic nonlinear Schrödinger equation, the authors extended the scattering results on radial H^1 solutions to the nonradial case . The technique employed is parallel to that employed by Kenig-Merle [9] in their study of the energy-critical NLS. For $0 < s_c < 1$, the author in [17] have extended the scattering results to the general L^2 supercritical and H^1 subcritical case.

Then in Holmer and Roudenko [7], the authors further studied the blow-up theory for the 3D cubic nonlinear Schrödinger equation , which dropped the additional hypothesis of finite variance and radiality. More precisely, they proved that if $M(u)E(u) < M(Q)E(Q)$ and $\|\nabla u_0\|_2 \|u_0\|_2 > \|\nabla Q\|_2 \|Q\|_2$, then either $u(t)$ blows up in finite positive time, or $u(t)$ exists globally for all positive time and there exists a time sequence $t_n \to +\infty$ such that $\|\nabla u(t_n)\|_2 \to \infty$, with similar results holding for negative time.

In this paper, we extend the above results to the general L^2 supercritical and H^1 subcritical case, and obtain the following conclusion:

Theorem 1.1. *Suppose* $u_0 \in H^1$, $M(u)^{\frac{1-s_c}{s_c}} E(u) < M(Q)^{\frac{1-s_c}{s_c}} E(Q)$ *and* $\|\nabla u_0\|_2 \|u_0\|_2^{\frac{1-s_c}{s_c}} > \|\nabla Q\|_2 \|Q\|_2^{\frac{1-s_c}{s_c}}$. *Then either* $u(t)$ *blows up in finite forward time, or* $u(t)$ *is forward global and there exists a time sequence* $t_n \to \infty$ *such that* $\|\nabla u(t_n)\|_2 \to \infty$. *A similar statement holds for negative time.*

Different from a similar result obtained by Glangeta and Merle [5] for the case $E(u) < 0$, our proof is by means of the profile decomposition introduced by Keraani [11], nonlinear pertubation theory based on the Strichartz estimate [1] [10], and the rigidity theorems based upon the localized virial identity. Though with the same idea as in [7], we still have to reestablish the tools mentioned above, such as the nonlinear profile decomposition, to conquer the difficulties our general case should bring .

Remark 1.2. Via the Galilean transform and momentum conservation, in this paper, we will always assume that $P(u) = 0$, and put further standard details in the Appendix. That is to say we need only show Theorem 1.1 under the condition $P(u) = 0$.

In this paper, we denote the Sobolev space $H^1(\mathbb{R}^N)$ as H^1 for short, and the L^p norm $\|\cdot\|_p$. Also for convenience, we will use the notation C, except for some specifications, standing for the variant absolute constants.

2. Preliminaries

In this section, we will review some basic facts about the ground state and give a dichotomy result.

Weinstein in [16] proved that the sharp constant C_{GN} of Gagliardo-Nirenberg inequality for $0 < s_c < 1$

$$\|u\|_{L^{p+1}(R^N)}^{p+1} \le C_{GN} \|\nabla u\|_{L^2(R^N)}^{\frac{N(p-1)}{2}} \|u\|_{L^2(R^N)}^{2-\frac{(N-2)(p-1)}{2}} \tag{2.1}$$

is achieved by $u = Q$, where Q is the ground state of

$$-(1-s_c)Q + \Delta Q + |Q|^{p-1}Q = 0.$$

Using Pohozhaev identities we can get the following identities without difficulty:

$$\|Q\|_2^2 = \frac{2}{N}\|\nabla Q\|_2^2,$$

$$\|Q\|_{p+1}^{p+1} = \frac{2(p+1)}{N(p-1)}\|\nabla Q\|_2^2 = \frac{(p+1)}{(p-1)}\|Q\|_2^2,$$

$$E(Q) = \frac{N(p-1)-4}{2N(p-1)}\|\nabla Q\|_2^2 = \frac{N(p-1)-4}{4(p-1)}\|Q\|_2^2 = \frac{N(p-1)-4}{4(p+1)}\|Q\|_{p+1}^{p+1}, \tag{2.2}$$

and C_{GN} can be expressed by

$$C_{GN} = \frac{\|Q\|_{p+1}^{p+1}}{\|\nabla Q\|_2^{\frac{N(p-1)}{2}} \|Q\|_2^{2-\frac{(N-2)(p-1)}{2}}}. \tag{2.3}$$

Note that the Sobolev $\dot{H}^{s_c}$ norm and the equation (1.1) are invariant under the scaling $u(x,t) \mapsto u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t)$. Other scaling invariant quantities are $\|\nabla u\|_2 \|u\|_2^{\frac{1-s_c}{s_c}}$ and $E(u)M(u)^{\frac{1-s_c}{s_c}}$.

Let

$$\eta(t) = \frac{\|\nabla u\|_2 \|u\|_2^{\frac{1-s_c}{s_c}}}{\|\nabla Q\|_2 \|Q\|_2^{\frac{1-s_c}{s_c}}}. \tag{2.4}$$

In order to study the relationship between $\eta(t)$ and $\frac{E(u)M(u)^{\frac{1-s_c}{s_c}}}{E(Q)M(Q)^{\frac{1-s_c}{s_c}}}$, we might as well assume $\|u\|_2 = \|Q\|_2$ by scaling. Denote $\omega_1 = \frac{N(p-1)}{N(p-1)-4}$ and $\omega_2 = \frac{4}{N(p-1)-4}$. Then by (2.1)-(2.3) we have

$$
\begin{aligned}
&2\omega_1 \frac{\|\nabla u\|_2^2 \|u\|_2^{\frac{2-2s_c}{s_c}}}{\|\nabla Q\|_2^2 \|Q\|_2^{\frac{2-2s_c}{s_c}}} \geq \frac{E(u)M(u)^{\frac{1-s_c}{s_c}}}{E(Q)M(Q)^{\frac{1-s_c}{s_c}}} = \frac{E(u)}{E(Q)} \\
&= \omega_1 \frac{\|\nabla u\|_2^2}{\|\nabla Q\|_2^2} - \frac{2\omega_1}{p+1} \frac{\|u\|_{p+1}^{p+1}}{\|Q\|_{p+1}^{p+1}} \\
&\geq \omega_1 \frac{\|\nabla u\|_2^2}{\|\nabla Q\|_2^2} - \frac{2\omega_1}{p+1} \frac{C_{GN} \|\nabla u\|_2^{\frac{N(p-1)}{2}} \|u\|_2^{2-\frac{(N-2)(p-1)}{2}}}{\|Q\|_2^2} \\
&= \omega_1 \eta(t)^2 - \frac{2\omega_1}{p+1} \frac{C_{GN} \|Q\|_2^{2-\frac{(N-2)(p-1)}{2}}}{\|\nabla Q\|_2^{2-\frac{(N-2)(p-1)}{2}}} \frac{\|\nabla u\|_2^{\frac{N(p-1)}{2}}}{\|\nabla Q\|_2^{\frac{N(p-1)}{2}}} \\
&= \omega_1 \eta(t)^2 - \frac{4\omega_1}{N(p-1)} \eta(t)^{\frac{N(p-1)}{2}} = \omega_1 \eta(t)^2 - \omega_2 \eta(t)^{\frac{N(p-1)}{2}}.
\end{aligned}
$$

That is

$$
2\omega_1 \eta(t)^2 \geq \frac{E(u)M(u)^{\frac{1-s_c}{s_c}}}{E(Q)M(Q)^{\frac{1-s_c}{s_c}}} \geq \omega_1 \eta(t)^2 - \omega_2 \eta(t)^{\frac{N(p-1)}{2}}. \tag{2.5}
$$

Note that $\frac{\omega_1}{\omega_2} > 1$ as $\frac{4}{N} < p-1 < \frac{4}{N-2}$. Thus it is not difficult to observe that if $0 \leq M(u)^{\frac{1-s_c}{s_c}} E(u)/M(Q)^{\frac{1-s_c}{s_c}} E(Q) < 1$, then there exist two solutions $0 \leq \lambda_- < 1 < \lambda$ to the following equation of λ

$$
\frac{E(u)M(u)^{\frac{1-s_c}{s_c}}}{E(Q)M(Q)^{\frac{1-s_c}{s_c}}} = \omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}}. \tag{2.6}
$$

By the H^1 local theory [1] , there exist $-\infty \leq T_- < 0 < T_+ \leq +\infty$ such that (T_-, T_+) is the maximal time interval of existence for $u(t)$ solving (1.1) , and if $T_+ < +\infty$ then

$$
\|\nabla u(t)\|_2 \geq \frac{C}{(T_+ - t)^{\frac{1}{p-1} - \frac{N-2}{4}}} \quad as\ t \uparrow T_+,
$$

and a similar argument holds if $-\infty < T_-$. Moreover, as a consequence of the continuity of the flow $u(t)$, we have the following dichotomy proposition :

Proposition 2.1. *(Global versus blow-up dichotomy) Let* $u_0 \in H^1(R^N)$, *and let* $I = (T_-, T_+)$ *be the maximal time interval of existence of* $u(t)$ *solving* (1.1). *Suppose that*

$$
M(u)^{\frac{1-s_c}{s_c}} E(u) < M(Q)^{\frac{1-s_c}{s_c}} E(Q). \tag{2.7}
$$

If (2.7) *holds and*

$$
\|u_0\|_2^{\frac{1-s_c}{s_c}} \|\nabla u_0\|_2 < \|Q\|_2^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2, \tag{2.8}
$$

then $I = (-\infty, +\infty)$, *i.e., the solution exists globally in time, and for all time* $t \in \mathbb{R}$,

$$
\|u(t)\|_2^{\frac{1-s_c}{s_c}} \|\nabla u(t)\|_2 < \|Q\|_2^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2. \tag{2.9}
$$

If (2.7) *holds and*

$$\|u_0\|_2^{\frac{1-s_c}{s_c}}\|\nabla u_0\|_2 > \|Q\|_2^{\frac{1-s_c}{s_c}}\|\nabla Q\|_2, \tag{2.10}$$

then for $t \in I$,

$$\|u(t)\|_2^{\frac{1-s_c}{s_c}}\|\nabla u(t)\|_2 > \|Q\|_2^{\frac{1-s_c}{s_c}}\|\nabla Q\|_2. \tag{2.11}$$

Proof. Multiplying the formula of energy by $M(u)^{\frac{1}{s_c}-1}$ and using the Gagliardo-Nirenberg inequality we have

$$\begin{aligned} E(u)M(u)^{\frac{1}{s_c}-1} &= \frac{1}{2}\|\nabla u\|_{L^2}^2\|u\|_{L^2}^{\frac{2}{s_c}-2} - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}\|u\|_{L^2}^{\frac{2}{s_c}-2} \\ &\geq \frac{1}{2}(\|\nabla u\|_2\|u\|_2^{\frac{1-s_c}{s_c}})^2 - \frac{1}{p+1}C_{GN}(\|\nabla u\|_2\|u\|_2^{\frac{1-s_c}{s_c}})^{\frac{N(p-1)}{2}}. \end{aligned}$$

Define $f(x) = \frac{1}{2}x^2 - \frac{1}{p+1}C_{GN}x^{\frac{N(p-1)}{2}}$. Since $N(p-1) \geq 4$, then $f'(x) = x(1 - C_{GN}\frac{N(p-1)}{2(p+1)}x^{\frac{N(p-1)-4}{2}})$, and $f'(x) = 0$ when $x_0 = 0$ and $x_1 = \|\nabla Q\|_2\|Q\|_2^{\frac{1-s_c}{s_c}}$. Note that $f(0) = 0$ and $f(x_1) = E(u)M(u)^{\frac{1}{s_c}-1}$, thus the graph of f has two extrema: a local minimum at x_0 and a local maximum at x_1. The condition (2.7) implies that $E(u_0)M(u_0)^{\frac{1}{s_c}-1} < f(x_1)$. Combining with energy conservation, we have

$$f(\|\nabla u\|_2\|u\|_2^{\frac{1-s_c}{s_c}}) \leq E(u)M(u_0)^{\frac{1}{s_c}-1} = E(u)M(u)^{\frac{1}{s_c}-1} < f(x_1). \tag{2.12}$$

If initially $\|\nabla u_0\|_2\|u_0\|_2^{\frac{1-s_c}{s_c}} < x_1$, i.e., the condition (2.8) holds, then by (2.12) and the continuity of $\|\nabla u(t)\|_2$ in t, we have $\|\nabla u(t)\|_2\|u(t)\|_2^{\frac{1-s_c}{s_c}} < x_1$ for all time $t \in I$. In particular, the H^1 norm of the solution is bounded, which implies the global existence and (2.9) in this case.

If initially $\|\nabla u_0\|_2\|u_0\|_2^{\frac{1-s_c}{s_c}} > x_1$, i.e., the condition (2.10) holds, then by (2.12) and the continuity of $\|\nabla u(t)\|_2$ in t, we have $\|\nabla u(t)\|_2\|u(t)\|_2^{\frac{1-s_c}{s_c}} > x_1$ for all time $t \in I$, which proves (2.11).

□

The following is another statement of the dichotomy proposition in terms of λ and $\eta(t)$ defined by (2.6) and (2.4) respectively, which will be useful in the sequel.

Proposition 2.2. *Let* $M(u)^{\frac{1-s_c}{s_c}}E(u) < M(Q)^{\frac{1-s_c}{s_c}}E(Q)$ *and* $0 \leq \lambda_- < 1 < \lambda$ *be defined by* (2.6). *Then exactly one of the following holds:*
(1) The solution $u(t)$ *to* (1.1) *is global and*

$$\frac{1}{2\omega_1}\frac{E(u)M(u)^{\frac{1-s_c}{s_c}}}{E(Q)M(Q)^{\frac{1-s_c}{s_c}}} \leq \eta(t)^2 \leq \lambda_-^2, \quad \forall\ t \in (-\infty, +\infty)$$

(2) $1 < \lambda^2 \leq \eta(t)^2$, $\forall\ t \in (T_-, T_+)$.

Naturally, whether the solution is of the first or second type in Proposition 2.2 is determined by checking the initial data. Note that the second case does not assert finite-time blow-up. In the first case, we have further results as follows , the proof of which is almost the same as [17].

Lemma 2.3. *(Small initial data). Let* $\|u_0\|_{\dot{H}^{s_c}} \leq A$*, then there exists* $\delta_{sd} = \delta_{sd}(A) > 0$ *such that if* $\|e^{it\Delta}u_0\|_{S(\dot{H}^{s_c})} \leq \delta_{sd}$*, then* u *solving* (1.1) *is global and*

$$\|u\|_{S(\dot{H}^{s_c})} \leq 2\|e^{it\Delta}u_0\|_{S(\dot{H}^{s_c})}, \tag{2.13}$$

$$\|D^{s_c}u\|_{S(L^2)} \leq 2c\|u_0\|_{\dot{H}^{s_c}}. \tag{2.14}$$

(one will find $\|\cdot\|_{S(\dot{H}^{s_c})}$ *in Section 6, and note that by Strichartz estimates, the hypotheses are satisfied if* $\|u_0\|_{\dot{H}^{s_c}} \leq C\delta_{sd}$*.)*

Lemma 2.4. *(Existence of wave operators). Suppose that* $\psi^+ \in H^1$ *and*

$$\frac{1}{2}||\nabla\psi^+||_2^2 M(\psi^+)^{\frac{1-s_c}{s_c}} < E(Q)M(Q)^{\frac{1-s_c}{s_c}}. \tag{2.15}$$

Then there exists $v_0 \in H^1$ *such that* v *solves* (1.1) *with initial data* v_0 *globally in* H^1 *with*

$$\|\nabla v(t)\|_2\|v_0\|_2^{\frac{1-s_c}{s_c}} < \|\nabla Q\|_2\|Q\|_2^{\frac{1-s_c}{s_c}}, M(v) = \|\psi^+\|_2^2, E[v] = \frac{1}{2}\|\nabla\psi^+\|_2^2,$$

and

$$\lim_{t\to+\infty} \|v(t) - e^{it\Delta}\psi^+\|_{H^1} = 0.$$

Moreover, if $\|e^{it\Delta}\psi^+\|_{S(\dot{H}^{s_c})} \leq \delta_{sd}$*, then*

$$\|v_0\|_{\dot{H}^{s_c}} \leq 2\|\psi^+\|_{\dot{H}^{s_c}} \text{ and } \|v\|_{S(\dot{H}^{s_c})} \leq 2\|e^{it\Delta}\psi^+\|_{S(\dot{H}^{s_c})}.$$

$$\|D^s v\|_{S(L^2)} \leq c\|\psi^+\|_{\dot{H}^s}, 0 \leq s \leq 1.$$

Theorem 2.5. *(Scattering). If* $0 < M(u)^{\frac{1-s_c}{s_c}}E(u)/M(Q)^{\frac{1-s_c}{s_c}}E(Q) < 1$ *and the first case of Proposition 2.2 holds, then* $u(t)$ *scatters as* $t \to +\infty$ *or* $t \to -\infty$*. That means there exist* $\phi_\pm \in H^1$ *such that*

$$\lim_{t\to\pm\infty} \|u(t) - e^{-it\Delta}\phi_\pm\|_{H^1} = 0. \tag{2.16}$$

Consequently, we have that

$$\lim_{t\to\pm\infty} \|u(t)\|_{L^{p+1}} = 0 \tag{2.17}$$

and

$$\lim_{t\to\pm\infty} \eta(t)^2 = \frac{1}{2\omega_1}\frac{E(u)M(u)^{\frac{1-s_c}{s_c}}}{E(Q)M(Q)^{\frac{1-s_c}{s_c}}}. \tag{2.18}$$

3. Virial Identity and Blow-Up Conditions

In the sequel we focus on the second case of Proposition 2.2. Using the classical virial identity, we first derive the upper bound on the finite blow-up time under the finite variance hypothesis.

Proposition 3.1. *Let $M(u) = M(Q)$, $E(u) < E(Q)$. Suppose $\|xu_0\|_2 < +\infty$ and suppose the second case of Proposition 2.2 holds ($\lambda > 1$ is defined by (2.6)).Define $r(t)$ to be the scaled variance:*

$$r(t) = \frac{\|xu\|_2^2}{\left(-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}\right) E(Q)}$$

Then blow-up occurs in forward time before t_b, where

$$t_b = r'(0) + \sqrt{r'(0)^2 + 2r(0)}.$$

Note that

$$r(0) = \frac{\|xu_0\|_2^2}{\left(-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}\right) E(Q)}$$

and

$$r'(0) = \frac{Im \int (x \cdot \nabla u_0)\overline{u_0}}{\left(-4\omega_1\lambda^2 + N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}\right) E(Q)}.$$

Proof. The virial identity gives

$$r''(t) = \frac{4N(p-1)E(u) - (2N(p-1)-8)\|\nabla u\|_2^2}{\left(-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}\right) E(Q)}.$$

Identities (2.2) imply

$$r''(t) = \frac{4N(p-1)\frac{E(u)}{E(Q)} - 2\omega_1\left(2N(p-1)-8\right)\frac{\|\nabla u\|_2^2}{\|\nabla Q\|_2^2}}{-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}}.$$

By the definition of λ and η,

$$r''(t) = \frac{4N(p-1)(\omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}) - 2\omega_1\left(2N(p-1)-8\right)\eta(t)^2}{-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}}.$$

Since $\eta(t) \geq \lambda > 1$, we have

$$r''(t) \leq -1,$$

which, by integrating in time twice, gives

$$r(t) \leq -\frac{1}{2}t^2 + r'(0)t + r(0).$$

The positive root of the polynomial on the right hand side is t_b given in the proposition statement.

□

The next result is related to the local virial identity. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be radial such that

$$\varphi(x) = \begin{cases} |x|^2, & |x| \leq 1; \\ 0, & |x| \geq 2. \end{cases}$$

For $R > 0$ define

$$z_R(t) = \int R^2 \phi(\frac{x}{R})|u(x,t)|^2 dx. \tag{3.1}$$

Then we can directly calculate the following local virial identity:

$$\begin{aligned} z_R''(t) &= 4\int \partial_j \partial_k \phi(\frac{x}{R})\partial_j u \partial_k \bar{u} dx - \int \Delta\phi(\frac{x}{R})|u|^4 dx - \frac{1}{R^2}\int \Delta^2\phi(\frac{x}{R})|u|^2 dx \\ &= \left(4N(p-1)E(u) - (2N(p-1)-8)\|\nabla u\|_2^2\right) + A_R(u(t)), \end{aligned} \tag{3.2}$$

where for a constant C_1 we can control

$$A_R(u(t)) \leq C_1\left(\frac{1}{R^2}\|u\|^2_{L^2(|x|\geq R)} + \|u\|^{p+1}_{L^{p+1}(|x|\geq R)}\right). \tag{3.3}$$

The local virial identity will give another version of Proposition 3.1 , for which , without the assumption of finite variance, we will assumes that the solution is suitably localized in H^1 for all times. Define

$$\eta_{\geq R} = \frac{\|u\|^{s_c(p-1)}_{L^2(|x|\geq R)}\|\nabla u\|^{(1-s_c)(p-1)}_{L^2(|x|\geq R)}}{\|Q\|_2^{s_c(p-1)}\|\nabla Q\|_2^{(1-s_c)(p-1)}}.$$

Proposition 3.2. *Let $M(u) = M(Q)$, $E(u) < E(Q)$ and suppose the second case of Proposition 2.2 holds ($\lambda > 1$ is defined in (2.6)). Select γ such that*

$$0 < \gamma < min\left(2\omega_1\left(2N(p-1)-8\right), 4N(p-1)\omega_2\lambda^{\frac{N(p-1)-4}{2}} - 16\omega_1\right).$$

Suppose that there is a radius $R \geq C_2\gamma^{-\frac{1}{2}}$ such that for all t, there holds $\eta_{\geq R} \leq \gamma$. Define $\tilde{r}(t)$ to be the scaled local variance:

$$\tilde{r}(t) = \frac{z_R(t)}{CE(Q)\left(-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}} - \gamma\lambda^2\right)}$$

(C is an absolute constant determined by C_1 and C_2). Then blow-up occurs in forward time before t_b, where

$$t_b = \tilde{r}'(0) + \sqrt{\tilde{r}'(0)^2 + 2\tilde{r}(0)}.$$

Proof. By the local virial identity and the same steps in the proof of Proposition 3.1

$$\tilde{r}''(t) = \frac{1}{C}\frac{4N(p-1)(\omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}) - 2\omega_1\left(2N(p-1)-8\right)\eta(t)^2 + A_R(u(t))/E(Q)}{-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}} - \gamma\lambda^2}$$

By the exterior Gagliardo-Nirenberg inequality, we have

$$\|u\|_{L^{p+1}(|x|\geq R)}^{p+1} \leq C_{GN}\|\nabla u\|_{L^2(|x|\geq R)}^{\frac{N(p-1)}{2}}\|u\|_{L^2(|x|\geq R)}^{2-\frac{(N-2)(p-1)}{2}} \leq \|\nabla u\|_2^2\eta_{\geq R} \leq \|\nabla Q\|_2^2\gamma\eta(t)^2. \tag{3.4}$$

This combined with

$$\frac{1}{R^2}\|u\|_{L^2(|x|\geq R)}^2 \leq C_2^{-2}\|Q\|_2^2\gamma \leq C_2^{-2}\|Q\|_2^2\gamma\eta(t)^2 \tag{3.5}$$

gives

$$\begin{aligned}\tilde{r}''(t) &\leq \frac{1}{C}\frac{4N(p-1)(\omega_1\lambda^2-\omega_2\lambda^{\frac{N(p-1)}{2}}) - 2\omega_1\left(2N(p-1)-8\right)\eta(t)^2 + C_3\frac{(\|Q\|_2^2+\|\nabla Q\|_2^2)}{E(Q)}\gamma\eta(t)^2}{-16\omega_1\lambda^2+4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}-\gamma\lambda^2} \\ &\leq \frac{1}{C}\frac{C_4\left(4N(p-1)(\omega_1\lambda^2-\omega_2\lambda^{\frac{N(p-1)}{2}}) - 2\omega_1\left(2N(p-1)-8\right)\eta(t)^2 + \gamma\eta(t)^2\right)}{-16\omega_1\lambda^2+4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}-\gamma\lambda^2}.\end{aligned}$$

Taking the constant $C = C_4$, since $\eta(t) \geq \lambda > 1$ and from the selection of γ, we obtain

$$r''(t) \leq -1.$$

The remainder of the argument is the same as in the proof of Proposition 3.1 .

□

Finally, we will give the finite blow-up time for radial solutions before which we would like to introduce the Radial Gagliardo-Nirenberb inequality:

Lemma 3.3. [15] *(Radial Gagliardo-Nirenberb inequality). For all $\delta > 0$, there exists a constant $C_\delta > 0$ such that for all $u \in \dot{H}^{s_c}$ with radial symmetry, and for all $R > 0$, we have*

$$\int_{|x|\geq R}|u|^{p+1}dx \leq \delta\int_{|x|\geq R}|\nabla u|^2dx + \frac{C_\delta}{R^{2(1-s_c)}}\left[(\rho(u,R))^{\frac{2(p+3)}{5-p}} + (\rho(u,R))^{\frac{p+1}{2}}\right],$$

where $\rho(u,R) = \sup_{R'\geq R}\frac{1}{(R')^{2s_c}}\int_{R'\leq|x|\leq 2R'}|u|^2dx$.

Note that this lemma implies that for all $\delta > 0$, there exists a constant $C_\delta > 0$ and $C_Q > 0$ such that for all $u \in \dot{H}^{s_c}$ with radial symmetry and $M(u) = M(Q)$, and for all $R > 0$, we have

$$\int_{|x|\geq R}|u|^{p+1}dx \leq \delta\int_{|x|\geq R}|\nabla u|^2dx + \frac{C_\delta C_Q}{R^{2(1-s_c)}}. \tag{3.6}$$

Proposition 3.4. *Let $M(u) = M(Q)$, $E(u) < E(Q)$ and suppose the second case of Proposition 2.2 holds ($\lambda > 1$ is defined in* (2.6)*.) Suppose that u is radial. Select γ such that*

$$0 < \gamma < min\left(2\omega_1\left(2N(p-1)-8\right), 4N(p-1)\omega_2\lambda^{\frac{N(p-1)-4}{2}} - 16\omega_1\right).$$

Then for

$$R > \max\left(\gamma^{-\frac{1}{2}}, \left(\frac{2C_\gamma}{-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}} - \gamma\lambda^2}\right)^{\frac{1}{2(1-s_c)}}\right)$$

we define $\tilde{r}(t)$ to be the scaled local variance:

$$\tilde{r}(t) = \frac{z_R(t)}{\tilde{C}_Q E(Q)\left(-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}} - \gamma\lambda^2\right)}$$

(where the constant $\tilde{C}_Q$ is dependent on Q determined by C_γ and C_Q in (3.6)*). Then blow-up occurs in forward time before t_b, where*

$$t_b = \tilde{r}'(0) + \sqrt{\tilde{r}'(0)^2 + 2\tilde{r}(0)}.$$

Proof. We modify the proof of Proposition 3.2 only in (3.4) and (3.5). From the Radial Gagliardo-Nirenberb inequality (3.6) with $\delta = \gamma$, we obtain

$$\|u\|^{p+1}_{L^{p+1}(|x|\geq R)} \leq C_Q\left(\gamma\eta(t)^2 + \frac{C_\gamma}{R^{2(1-s_c)}}\right).$$

If taking C_Q to stand for the variant constants dependent on Q, we have

$$\frac{1}{R^2}\|u\|^2_{L^2(|x|\geq R)} \leq \frac{C_Q}{R^2} \leq \frac{C_Q\eta(t)^2}{R^2} \leq C_Q\gamma\eta(t)^2.$$

Thus

$$\begin{aligned}
\tilde{r}''(t) &\leq C_Q\frac{4N(p-1)(\omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}) - 2\omega_1\left(2N(p-1) - 8\right)\eta(t)^2 + \gamma\eta(t)^2 + \frac{C_\gamma}{R^{2(1-s_c)}}}{\tilde{C}_Q\left(-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}} - \gamma\lambda^2\right)} \\
&\leq C_Q\frac{\left(4N(p-1)(\omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}) - 2\omega_1\left(2N(p-1) - 8\right)\eta(t)^2 + \gamma\eta(t)^2\right) + \frac{C_\gamma}{R^{2(1-s_c)}}}{\tilde{C}_Q\left(-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}} - \gamma\lambda^2\right)}.
\end{aligned}$$

We only have to select $\tilde{C}_Q = 2C_Q$ in the assumptions. Then since $\eta(t) \geq \lambda > 1$, the restriction of γ and R gives

$$r''(t) \leq -1,$$

and we conclude the proof with the same steps as in the proof of Proposition 3.1. □

4. Variational Characterization of the Ground State

This section deals with the variation characterization of Q stated in the above section. It is an important preparation for the "near boundary case" in Section 5. For now, we will write $u = u(x)$ as the time dependence plays no role in what follows.

Proposition 4.1. *There exists a function $\epsilon(\rho)$ with $\epsilon(\rho) \to 0$ as $\rho \to 0$ such that the following holds: suppose there is $\lambda > 0$ satisfying*

$$\left| \frac{M(u)^{\frac{1-s_c}{s_c}} E(u)}{M(Q)^{\frac{1-s_c}{s_c}} E(Q)} - \left(\omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}} \right) \right| \leq \rho \lambda^{\frac{N(p-1)}{2}}, \tag{4.1}$$

and

$$\left| \frac{\|u\|_2^{\frac{1-s_c}{s_c}} \|\nabla u\|_2}{\|Q\|_2^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2} - \lambda \right| \leq \rho \begin{cases} \lambda, & \lambda \geq 1 \\ \lambda^2, & \lambda \leq 1. \end{cases} \tag{4.2}$$

Then there exists $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$ such that

$$\left\| u - e^{i\theta} \lambda^{\frac{N}{2}} \beta^{-\frac{2}{p-1}} Q\left(\lambda(\beta^{-1} \cdot - x_0) \right) \right\|_2 \leq \beta^{\frac{N}{2} - \frac{2}{p-1}} \epsilon(\rho) \tag{4.3}$$

and

$$\left\| \nabla \left[u - e^{i\theta} \lambda^{\frac{N}{2}} \beta^{-\frac{2}{p-1}} Q\left(\lambda(\beta^{-1} \cdot - x_0) \right) \right] \right\|_2 \leq \lambda \beta^{\frac{N}{2} - \frac{2}{p-1} - 1} \epsilon(\rho), \tag{4.4}$$

where $\beta = (\frac{M(u)}{M(Q)})^{\frac{p-1}{N(p-1)-4}}$.

Remark 4.2. If we let $v(x) = \beta^{\frac{2}{p-1}} u(\beta x)$, then $M(v) = \beta^{\frac{4}{p-1} - N} M(u) = M(Q)$, and we can then restate Proposition 4.1 as follows:

Suppose $\|v\|_2 = \|Q\|_2$ and there is $\lambda > 0$ such that

$$\left| \frac{E(v)}{E(Q)} - \left(\omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}} \right) \right| \leq \rho \lambda^{\frac{N(p-1)}{2}}, \tag{4.5}$$

and

$$\left| \frac{\|\nabla v\|_2}{\|\nabla Q\|_2} - \lambda \right| \leq \rho \begin{cases} \lambda, & \lambda \geq 1 \\ \lambda^2, & \lambda \leq 1. \end{cases} \tag{4.6}$$

Then there exists $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$ such that

$$\left\| v - e^{i\theta} \lambda^{\frac{N}{2}} Q\left(\lambda(\cdot - x_0) \right) \right\|_2 \leq \epsilon(\rho) \tag{4.7}$$

and

$$\left\| \nabla \left[v - e^{i\theta} \lambda^{\frac{N}{2}} Q\left(\lambda(\cdot - x_0) \right) \right] \right\|_2 \leq \lambda \epsilon(\rho). \tag{4.8}$$

Thus it suffices to prove the scaled statement equivalent to Proposition 4.1 and we will carry it out by means of the following result from Lions [13].

Proposition 4.3. *There exists a function $\epsilon(\rho)$, defined for small $\rho > 0$ such that* $\lim_{\rho \to 0} \epsilon(\rho) = 0$, *such that for all $u \in H^1$ with*

$$|\|u\|_{p+1} - \|Q\|_{p+1}| + |\|u\|_2 - \|Q\|_2| + |\|\nabla u\|_2 - \|\nabla Q\|_2| \leq \rho, \tag{4.9}$$

there exist $\theta_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$ such that

$$\left\| u - e^{i\theta_0} Q(\cdot - x_0) \right\|_{H^1} \leq \epsilon(\rho). \tag{4.10}$$

Proof. (Proof of proposition 4.1). As a result of Remark 4.2, we will just prove the equivalent version rescaling off the mass. Set $\tilde{u}(x) = \lambda^{-\frac{N}{2}} v(\lambda^{-1}x)$, and then (4.6) gives

$$\left| \frac{\|\nabla \tilde{u}\|_2}{\|\nabla Q\|_2} - 1 \right| \leq \rho. \tag{4.11}$$

On the other hand, by (2.2) and the notation of ω_1 and ω_2 we have

$$\begin{aligned} \left| \frac{\|v\|_{p+1}^{p+1}}{\|Q\|_{p+1}^{p+1}} - \lambda^{\frac{N(p-1)}{2}} \right| &\leq \left| -\frac{1}{\omega_2} \left(\frac{E(v)}{E(Q)} - (\omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}}) \right) \right| + \left| \frac{N(p-1)}{4} \frac{\|\nabla v\|_2^2}{\|\nabla Q\|_2^2} - \frac{\omega_1}{\omega_2} \lambda^2 \right| \\ &= \frac{1}{\omega_2} \left| \frac{E(v)}{E(Q)} - (\omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}}) \right| + \frac{N(p-1)}{4} \left| \frac{\|\nabla v\|_2^2}{\|\nabla Q\|_2^2} - \lambda^2 \right|. \end{aligned}$$

Then (4.5) and (4.6) imply

$$\begin{aligned} \left| \frac{\|v\|_{p+1}^{p+1}}{\|Q\|_{p+1}^{p+1}} - \lambda^{\frac{N(p-1)}{2}} \right| &\leq \frac{1}{\omega_2} \rho \lambda^{\frac{N(p-1)}{2}} + \frac{N(p-1)}{4} \rho \begin{cases} \lambda^2, & \lambda \geq 1 \\ \lambda^4, & \lambda \leq 1 \end{cases} \\ &\leq \left(\frac{N(p-1)}{2} - 1\right) \rho \lambda^{\frac{N(p-1)}{2}}. \end{aligned}$$

Thus in terms of $\tilde{u}$, we obtain

$$\left| \frac{\|\tilde{u}\|_{p+1}^{p+1}}{\|Q\|_{p+1}^{p+1}} - 1 \right| \leq \frac{N(p-1) - 2}{2} \rho. \tag{4.12}$$

Thus (4.11) and (4.12) imply that the condition (4.9) is satisfied by $\tilde{u}$. By Proposition 4.3 and rescaling back to v, we obtain (4.7) and (4.8) .

□

5. Near-Boundary Case

We know from Proposition 2.2 that if $M(u) = M(Q)$ and $E(u)/E(Q) = \omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}}$ for some $\lambda > 1$ and $\|\nabla u_0\|_2 / \|\nabla Q\|_2 \geq \lambda$, then $\|\nabla u(t)\|_2 / \|\nabla Q\|_2 \geq \lambda$ for all t. Now in this section, we will claim that $\|\nabla u(t)\|_2 / \|\nabla Q\|_2$ cannot remain near λ globally in time.

Proposition 5.1. *Let $\lambda_0 > 1$. There exists $\rho_0 = \rho_0(\lambda_0) > 0$ with the property that $\rho_0(\lambda_0) \to 0$ as $\lambda_0 \to 1$, such that for any $\lambda \geq \lambda_0$, the following holds: There does not exist a solution $u(t)$ of problem* (1.1) *with $P(u) = 0$ satisfying $M(u) = M(Q)$,*

$$\frac{E(u)}{E(Q)} = \omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}}, \tag{5.1}$$

and for all $t \geq 0$

$$\lambda \leq \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2} \leq \lambda(1 + \rho_0). \tag{5.2}$$

We would like to give another equivalent statement implied by this assertion: For any solution $u(t)$ to (1.1) with $P(u)=0$ satisfying $M(u)=M(Q)$,

$$\frac{E(u)}{E(Q)}=\omega_1\lambda^2-\omega_2\lambda^{\frac{N(p-1)}{2}},$$

and for all $t\geq 0$

$$\lambda\leq\frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2},$$

there exist a time $t_0\geq 0$ such that

$$\frac{\|\nabla u(t_0)\|_2}{\|\nabla Q\|_2}\geq\lambda(1+\rho_0).$$

Before proving Proposition 5.1 we will firstly give a useful lemma the proof of which will be found in [7].

Lemma 5.2. *Suppose that $u(t)$ with $P(u)=0$ solving (1.1) satisfies, for all t*

$$\left\|u(t)-e^{i\theta(t)}Q(\cdot-x(t))\right\|_{H^1}\leq\epsilon \tag{5.3}$$

for some continuous functions $\theta(t)$ and $x(t)$. Then

$$\frac{|x(t)|}{t}\leq C\epsilon^2\quad \textit{as}\quad t\to+\infty.$$

Proof. (Proof of proposition 5.1). To the contrary, we suppose that there exists a solution $u(t)$ satisfying $M(u)=M(Q)$, $E(u)/E(Q)=\omega_1\lambda^2-\omega_2\lambda^{\frac{N(p-1)}{2}}$ and

$$\lambda\leq\frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2}\leq\lambda(1+\rho_0). \tag{5.4}$$

Since $\|\nabla u(t)\|_2^2\geq\lambda^2\|\nabla Q\|_2^2=2\omega_1\lambda^2E(Q)$, we have

$$\begin{aligned}&4N(p-1)E(u)-(2N(p-1)-8)\,\|\nabla u\|_2^2\\ \leq&-4\left(N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}-4\omega_1\lambda^2\right)E(Q).\end{aligned}$$

By Proposition 4.1, there exist functions $\theta(t)$ and $x(t)$ such that for $\rho=\rho_0$

$$\left\|u(t)-e^{i\theta(t)}\lambda^{\frac{N}{2}}Q\left(\lambda(\cdot-x(t))\right)\right\|_2\leq\epsilon(\rho) \tag{5.5}$$

and

$$\left\|\nabla\left[u(t)-e^{i\theta(t)}\lambda^{\frac{N}{2}}Q\left(\lambda(\cdot-x(t))\right)\right]\right\|_2\leq\lambda\epsilon(\rho). \tag{5.6}$$

By the continuity of the $u(t)$ flow, we may assume $\theta(t)$ and $x(t)$ are continuous. Let

$$R(T)=\max\left(\max_{0\leq t\leq T}|x(t)|,\log\epsilon(\rho)^{-1}\right).$$

For fixed T, take $R = R(T)$ in the local virial identity (3.2) . Then owing to the exponential localization of $Q(x)$, (5.5) and (5.6) imply that,

$$|A_R(u(t))| \leq \frac{C}{2}\lambda^2 \left(\epsilon(\rho) + e^{-R(T)}\right)^2 \leq C\lambda^2\epsilon(\rho)^2.$$

Taking $\rho = \rho_0$ small enough to make $\epsilon(\rho)$ small such that for all $0 \leq t \leq T$,

$$z_R''(t) \leq -CE(Q)(\lambda^{\frac{N(p-1)}{2}} - \lambda^2),$$

and so

$$\frac{z_R(T)}{T^2} \leq \frac{z_R(0)}{T^2} + \frac{z_R'(0)}{T} - CE(Q)(\lambda^{\frac{N(p-1)}{2}} - \lambda^2).$$

By definition of $z_R(t)$ we have

$$|z_R(0)| \leq CR^2\|u_0\|_2^2 = C\|Q\|_2^2R^2$$

and

$$|z_R'(0)| \leq CR\|u_0\|_2\|\nabla u_0\|_2 \leq C\|Q\|_2\|\nabla Q\|_2R(1+\rho_0)\lambda.$$

Consequently,

$$\frac{z_{2R(T)}(T)}{T^2} \leq C\left(\frac{R(T)^2}{T^2} + \frac{\lambda R(T)}{T}\right) - CE(Q)(\lambda^{\frac{N(p-1)}{2}} - \lambda^2).$$

Taking T sufficiently large ,Lemma 5.2 implies

$$0 \leq \frac{z_{2R(T)}(T)}{T^2} \leq C\left(\lambda\epsilon(\rho)^2 - (\lambda^{\frac{N(p-1)}{2}} - \lambda^2)\right) < 0$$

provided taking ρ_0 small enough .

Note that ρ_0 is independent of T . We then get a contradiction.

$\square$

6. Profile Decomposition

In this section we make some extension of the cubic profile decomposition [7] to our general case, and we review some work done by the author in [17].

First of all , we introduce some notations. We say that (q,r) is $\dot{H}^s(\mathbb{R}^N)$ admissible and denote it by $(q,r) \in \Lambda_s$ if

$$\frac{2}{q} + \frac{N}{r} = \frac{N}{2} - s, \quad \frac{2N}{N-2s} < r < \frac{2N}{N-2}$$

Correspondingly, we denote (q',r') the dual $\dot{H}^s(\mathbb{R}^N)$ admissible by $(q',r') \in \Lambda_s'$ if $(q,r) \in \Lambda_{-s}$ with (q',r') is the Hölder dual to (q,r). We also define the following Srichartz norm

$$\|u\|_{S(\dot{H}^s)} = \sup_{(q,r)\in\Lambda_s} \|u\|_{L_t^qL_x^r}$$

and the dual Strichartz norm

$$\|u\|_{S'(\dot{H}^{-s})} = \inf_{(q',r')\in\Lambda_s'} \|u\|_{L_t^{q'}L_x^{r'}} = \inf_{(q,r)\in\Lambda_{-s}} \|u\|_{L_t^{q'}L_x^{r'}},$$

where (q', r') is the Hölder dual to (q, r). Also as in [7] , the notation $S(\dot{H}^s; I)$ and $S'(\dot{H}^s; I)$ indicate a restriction to a time subinterval $I \subset (-\infty, +\infty)$.

Remark 6.1. By notation $\|\cdot\|_{S(\dot{H}^{s_c})}$ in the sequel, we will in fact add the restriction $q \geq r$ to the definition of $(q, r) \in \Lambda_{s_c}$ without affecting the future arguments for our main results in this paper, which is needed in the proof of Lemma 6.7 below.

Now we first restate the linear profile decomposition below which was shown in [17] .

Lemma 6.2. *(Profile expansion). Let $\phi_n(x)$ be an uniformly bounded sequence in H^1, then for each M there exists a subsequence of ϕ_n, also denoted by ϕ_n, and (1) for each $1 \leq j \leq M$, there exists a (fixed in n) profile $\tilde{\psi}^j(x)$ in H^1, (2) for each $1 \leq j \leq M$, there exists a sequence(in n)of time shifts t_n^j, (3) for each $1 \leq j \leq M$, there exists a sequence (in n) of space shifts x_n^j, (4) there exists a sequence (in n) of remainders $\tilde{W}_n^M(x)$ in H^1, such that*

$$\phi_n(x) = \sum_{j=1}^{M} e^{-it_n^j \Delta} \tilde{\psi}^j(x - x_n^j) + \tilde{W}_n^M(x),$$

The time and space sequences have a pairwise divergence property, i.e., for $1 \leq j \neq k \leq M$, we have

$$\lim_{n\to+\infty} (|t_n^j - t_n^k| + |x_n^j - x_n^k|) = +\infty.$$

The remainder sequence has the following asymptotic smallness property:

$$\lim_{M\to+\infty} [\lim_{n\to+\infty} \|e^{it\Delta} \tilde{W}_n^M\|_{S(\dot{H}^{s_c})}] = 0.$$

For fixed M and any $0 \leq s \leq 1$, we have the asymptotic Pythagorean expansion:

$$\|\phi_n\|_{\dot{H}^s}^2 = \sum_{j=1}^{M} \|\tilde{\psi}^j\|_{\dot{H}^s}^2 + \|\tilde{W}_n^M\|_{\dot{H}^s}^2 + o_n(1).$$

Remark 6.3. We omit the proof of Lemma 6.2, but would like to point out some modification from the statement in [17]: In the reference the author introduced a concept of k-point $(\frac{1}{r}, \frac{1}{q})$ for which one can also refer to [8], and gave the proof in terms of that conception. In fact, it is easy to check that $(\frac{1}{r}, \frac{1}{q})$ is a p-point with the same p in our equation (1.1) , if and only if $(q, r) \in \Lambda_{s_c}$. Moreover, it is interesting to note that if $(q, r) \in \Lambda_{s_c}$ then $(\frac{q}{p}, \frac{r}{p}) \in \Lambda'_{s_c}$. Thus we have the following Strichartz estimate which was frequently used in [17]:

$$\left\| i \int_0^t e^{i(t-t')\Delta} |u|^{p-1} |u|(x, t') dt' \right\|_{L_t^q L_x^r} \leq C \left\| |u|^{p-1} |u| \right\|_{L_t^{\frac{q}{p}} L_x^{\frac{r}{p}}} \leq C \left\| u \right\|_{L_t^q L_x^r}^p .$$

Furthermore, the author in [17] gave another useful claim and we will restate the equivalent version as follows: For any $(q, r) \in \Lambda_{s_c}$, there exists $(q_1, r_1) \in \Lambda_0$ and $(q_2', r_2') \in \Lambda_0'$ such that

$$\begin{cases} \frac{1}{q_2'} = \frac{1}{q_1} + \frac{p-1}{q} \\ \frac{1}{r_2'} = \frac{1}{r_1} + \frac{p-1}{r}. \end{cases}$$

Applying the above observation, our proof of Lemma 6.2 will be almost the same as that in [17], and that is why we will omit it here.

Similar to Keraani [11] and [9], we give the following definition of the nonlinear profile:

Definition 6.4. Let V be a solution to the linear Schrödinger equation. We say U is the nonlinear profile associated to $(V, \{t_n\})$, if U is a solution to the Hartree equation (1.1) satisfying

$$\|(U-V)(-t_n)\|_{H^1} \to 0 \quad as \quad n \to \infty.$$

Note that, similar to the arguments in [9], by the local theory and the proof of the existence of wave operators, there always exist a nonlinear profile associated to a given $(V, \{t_n\})$. Thus for every j, there exists a solution v^j to (1.1) associated to $(\tilde{\psi}^j, \{t_n^j\})$ such that

$$\|v^j(\cdot - x_n^j, -t_n^j) - e^{-it_n^j \Delta} \tilde{\psi}^j(\cdot - x_n^j)\|_{H^1} \to 0 \quad as \quad n \to \infty.$$

If we let $NLH(t)\psi$ denote the solution to (1.1) with initial data ψ, by shifting the linear profile $\tilde{\psi}^j$ when necessary, we may denote $v^j(-t_n^j)$ as $NLH(-t_n^j)\psi^j$ with some $\psi^j \in H^1$. Thus using the same method of replacing linear flows by nonlinear flows as applied in [6] to give the following proposition:

Proposition 6.5. *Let $\phi_n(x)$ be an uniformly bounded sequence in H^1 . There exists a subsequence of ϕ_n , also denoted by ϕ_n , profiles $\psi^j(x)$ in H^1 , and parameters x_n^j , t_n^j so that for each M ,*

$$\phi_n(x) = \sum_{j=1}^{M} NLS(-t_n^j)\psi^j(x - x_n^j) + W_n^M(x), \tag{6.1}$$

where as $n \to \infty$
- *For each j , either $t_n^j = 0$, $t_n^j \to +\infty$ or $t_n^j \to -\infty$.*
- *If $t_n^j \to +\infty$, then $\|NLS(-t)\psi^j\|_{S(\dot{H}^{s_c};[0,\infty))} < \infty$ and if $t_n^j \to -\infty$, then $\|NLS(-t)\psi^j\|_{S(\dot{H}^{s_c};[-\infty,0))} < \infty$*
- *For $j \neq k$,*
$$\lim_{n\to+\infty}(|t_n^j - t_n^k| + |x_n^j - x_n^k|) = +\infty.$$
- *$NLS(t)W_n^M$ is global for M large enough with*
$$\lim_{M\to+\infty}[\lim_{n\to+\infty} \|NLS(t)W_n^M\|_{S(\dot{H}^{s_c})}] = 0.$$

We also have the H^s Pythagorean decomposition: for fixed M and $0 \leq s \leq 1$,

$$\|\phi_n\|_{\dot{H}^s}^2 = \sum_{j=1}^{M} \|NLS(-t_n^j)\psi^j\|_{\dot{H}^s}^2 + \|W_n^M\|_{\dot{H}^s}^2 + o_n(1), \tag{6.2}$$

and the energy Pythagorean decomposition

$$E(\phi_n) = \sum_{j=1}^{M} E(\psi^j) + E(W_n^M) + o_n(1). \tag{6.3}$$

From a similar argument in [6], we know that (6.3) was proven by establishing the following first

$$\|\phi_n\|_{p+1}^{p+1} = \sum_{j=1}^{M} \|NLS(-t_n^j)\psi^j\|_{p+1}^{p+1} + \|W_n^M\|_{p+1}^{p+1} + o_n(1). \tag{6.4}$$

The next lemma is an extension of the perturbation theory for the case $N = 3$ [6]. By virtue of Remark 6.3, the proof will also be similar to [17], which we will represent in this paper.

Lemma 6.6. *(Perturbation Theory). For each $A \geq 1$, there exists $\epsilon_0 = \epsilon_0(A) \ll 1$ and $c = c(A)$ such that the following holds: Fix $T > 0$. Let $u = u(x,t) \in L^\infty([0,T]; H^1)$ solve*

$$iu_t + \Delta u + |u|^{p-1}u = 0$$

on $[0,T]$. Let $\tilde{u} = \tilde{u}(x,t) \in L^\infty([0,T]; H^1)$ and define

$$e = i\tilde{u}_t + \Delta\tilde{u} + |\tilde{u}|^{p-1}\tilde{u}.$$

For each $\epsilon \leq \epsilon_0$, if for some $(q_1, r_1) \in \Lambda_{-s_c}$

$$\|\tilde{u}\|_{S(\dot{H}^{s_c};[0,T])} \leq A, \quad \|e\|_{S'(\dot{H}^{-s_c};[0,T])} \leq \epsilon, \quad \text{and} \quad \|e^{it\Delta}(u(0) - \tilde{u}(0))\|_{S(\dot{H}^{s_c};[0,T])} \leq \epsilon,$$

then

$$\|u - \tilde{u}\|_{S(\dot{H}^{s_c};[0,T])} \leq c(A)\epsilon.$$

Proof. Under the condition of the lemma, it suffices to prove that for any $(q,r) \in \Lambda_{s_c}$ and for some $(q_1, r_1) \in \Lambda_{-s_c}$, if

$$\|\tilde{u}\|_{L_t^q L_x^r} \leq A, \quad \|e\|_{L_t^{q_1'} L_x^{r_1'}} \leq \epsilon, \quad \text{and} \quad \|e^{it\Delta}(u(0) - \tilde{u}(0))\|_{L_t^q L_x^r} \leq \epsilon,$$

then

$$\|u - \tilde{u}\|_{L_t^q L_x^r} \leq c(A)\epsilon.$$

In fact, the following arguments are similar to that in [17] except for some slight differences. One can also refer to [6] for a similar proof.

Let w defined by $u = \tilde{u} + w$, then w solves

$$iw_t + \Delta w + |\tilde{u} + w|^{p-1}(\tilde{u} + w) - |\tilde{u}|^{p-1}\tilde{u} + e = 0. \tag{6.5}$$

Since $\|\tilde{u}\|_{L_t^q L_x^r} \leq A$, we can partition $[0,T]$ into $N = N(A)$ intervals $I_j = [t_j, t_{j+1}]$ such that for every j, $\|\tilde{u}\|_{L_{t\in I_j}^q L_x^r} \leq \delta$ with δ sufficiently small to be specified later. The integral equation of (6.5) with initial data $w(t_j)$ is

$$w(t) = e^{i(t-t_j)\Delta}w(t_j) + i\int_{t_j}^{t} e^{i(t-s)\Delta}W(\cdot, s)ds, \tag{6.6}$$

where

$$W = (-|\tilde{u} + w|^{p-1}(\tilde{u} + w) + |\tilde{u}|^{p-1}\tilde{u}) - e.$$

Applying the inhomogeneous Strichartz estimate in I_j and from Remark 6.3, we have

$$\begin{aligned}
\|w\|_{L^q_{t\in I_j}L^r_x} &\leq \|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_{t\in I_j}L^r_x} \\
&\quad + C\|(-|\tilde{u}+w|^{p-1}(\tilde{u}+w)+|\tilde{u}|^{p-1}\tilde{u}\|_{L^{\frac{q}{p}}_{t\in I_j}L^{\frac{r}{p}}_x} + \|e\|_{L^{q_1}_t L^{r_1}_x} \\
&\leq \|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_{t\in I_j}L^r_x} \\
&\quad + C\|(|\tilde{u}|^{p-1}+|w|^{p-1})w\|_{L^{\frac{q}{p}}_{t\in I_j}L^{\frac{r}{p}}_x} + \|e\|_{L^{q_1}_t L^{r_1}_x} \\
&\leq \|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_{t\in I_j}L^r_x} \\
&\quad + C\|\tilde{u}\|^{p-1}_{L^q_{t\in I_j}L^r_x}\|w\|_{L^q_{t\in I_j}L^r_x} + C\|w\|^p_{L^q_{t\in I_j}L^r_x} + \|e\|_{L^{q_1}_t L^{r_1}_x}, \\
&\leq \|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_{t\in I_j}L^r_x} \\
&\quad + C\delta^{p-1}\|w\|_{L^q_{t\in I_j}L^r_x} + C\|w\|^p_{L^q_{t\in I_j}L^r_x} + C\epsilon.
\end{aligned}$$

If

$$\delta \leq \left(\frac{1}{4C}\right)^{\frac{1}{p-1}},\ \left(\|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_{t\in I_j}L^r_x} + C\epsilon_0\right) \leq \frac{1}{2}\left(\frac{1}{4C}\right)^{\frac{1}{p-1}}, \tag{6.7}$$

then

$$\|w\|_{L^q_{t\in I_j}L^r_x} \leq 2\|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_{t\in I_j}L^r_x} + 2C\epsilon.$$

Now we take $t = t_{j+1}$ in (6.6), and apply $e^{i(t-t_j)\Delta}$ to the both sides , we obtain

$$e^{i(t-t_{j+1})\Delta}w(t_{j+1}) = e^{i(t-t_j)\Delta}w(t_j) + i\int_{t_j}^{t_{j+1}} e^{i(t-s)\Delta}W(\cdot,s)ds.$$

Again, with the same method as above, we obtain

$$\|e^{i(t-t_{j+1})\Delta}w(t_{j+1})\|_{L^q_{t\in I_j}L^r_x} \leq 2\|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_{t\in I_j}L^r_x} + 2C\epsilon.$$

Iterating the above procedure from $j = 0$, we have

$$\|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_tL^r_x} \leq 2^j\|e^{i(t-t_0)\Delta}w(t_0)\|_{L^q_tL^r_x} + (2^j-1)2C\epsilon \leq 2^{j+2}C\epsilon.$$

To accommodate the second part of (6.7) for all intervals $I_j,\ 0 \leq j \leq N-1$, we require that

$$2^{N+2}C\epsilon_0 \leq \left(\frac{1}{4C}\right)^{\frac{1}{p-1}}, \tag{6.8}$$

and we obtain the result easily.

Now we recall the parameter dependence of parameters: We choose δ to meet the first part of (6.7). Given A, the number of the interval N is determined, and the inequality (6.8) tells how small ϵ_0 should be taken in terms of $N(A)$.

□

Note from the proof above that, the parameters in Lemma 6.6 is not dependent on T . As is stated in [7] for $N = 3$, besides the H^1 asymptotic orthogonality (6.2) at $t = 0$, this property can be extended to the NLS flow for $0 \le t \le T$ as an application of Lemma 6.6 with an constant $A = A(T)$ dependent on T (but only through A). As for the general Mass-supercritical and Energy-subcritical case, we can prove the following similar result:

Lemma 6.7. *(H^1 Pythagorean Decomposition Along the NLS Flow). Suppose $\phi_n(x)$ be a uniformly bounded sequence in H^1 . Fix any time $0 < T < \infty$. Suppose that $u_n(t) \equiv NLS(t)\phi_n$ exists up to time T for all n and*

$$\lim_{n\to\infty} \|\nabla u_n(t)\|_{L^\infty([0,T];L^2)} < \infty.$$

Let $W_n^M(t) \equiv NLS(t)W_n^M$. Then, for all j , $v^j(t) \equiv NLS(t)\psi^j$ exist up to time T and for all $t \in [0, T]$,

$$\|\nabla u_n\|_2^2 = \sum_{j=1}^{M} \|\nabla v^j(t - t_n^j)\|_2^2 + \|\nabla W_n^M(t)\|_2^2 + o_n(1). \tag{6.9}$$

Here, $o_n(1) \to 0$ uniformly on $0 \le t \le T$.

Proof. Let M_0 be such that for $M_1 \ge M_0$ and for δ_{sd} in Lemma 2.3, we have

$$\|NLS(t)W_n^{M_1}\|_{S(\dot{H}^{s_c})} \le \delta_{sd}/2$$

and $\|v^j\|_{S(\dot{H}^{s_c})} \le \delta_{sd}$ for $j > M_0$. Reorder the first M_0 profiles and introduce an index M_2 , $0 \le M_2 \le M_0$, such that

- For each $0 \le j \le M_2$ we have $t_n^j = 0$.(There is no j in this category if $M_2 = 0$.)
- For each $M_2+1 \le j \le M_0$ we have $|t_n^j| \to \infty$. (There is no j in this category if $M_2 = M_0$.)

By definition of M_0, $v^j(t)$ for $j > M_0$ scatters in both time directions. We claim that for fixed T and $M_2 + 1 \le j \le M_0$, $\|v^j(t - t_n^j)\|_{S(\dot{H}^{s_c};[0,T])} \to 0$ as $n \to \infty$. Indeed, take the case $t_n^j \to +\infty$ for example. By Proposition 6.5, $\|v^j(-t)\|_{S(\dot{H}^{s_c};[0,\infty))} < \infty$. Then for $q < \infty$, $\|v^j(-t)\|_{L^q([0,\infty);L^r)} < \infty$ implies $\|v^j(t-t_n^j)\|_{L^q([0,T];L^r)} \to 0$. On the other hand, since $v^j(t)$ in Proposition 6.5 is constructed by the existence of wave operators which converge in H^1 to a linear flow at $-\infty$, then the $L^{\frac{2N}{N-2s_c}}$ decay of the linear flow implies immediately that $\|v^j(t - t_n^j)\|_{L^\infty([0,T];L^{\frac{2N}{N-2s_c}})} \to 0$.

Let $B = \max(1, \lim_n \|\nabla u_n\|_{L^\infty([0,T];L^2)})$. For each $1 \le j \le M_2$, define $T^j \le T$ to be the maximal forward time on which $\|\nabla v^j\|_{L^\infty([0,T^j];L^2)} \le 2B$. Let $\widetilde{T} = \min_{1\le j\le M_2} T^j$, and if $M_2 = 0$, we just take $\widetilde{T} = T$. Note that if we have proved (6.9) holds for $T = \widetilde{T}$, then by definition of T^j , using the continuity arguments, it follows from (6.9) that for each $1 \le j \le M_2$, we have $T^j = T$. Hence $\widetilde{T} = T$. Thus, for the remainder of the proof, we just work on $[0, \widetilde{T}]$.

For each $1 \le j \le M_2$, $\|v^j\|_{L^\infty([0,\widetilde{T}];L^2)} = \|\psi^j\|_2 \le \lim_n \|\phi_n\|_2$ by (6.2) . Now, in view of the notation of $S(\dot{H}^{s_c}; [0, \widetilde{T}])$ and Remark 6.1, we will give the $S(\dot{H}^{s_c}; [0, \widetilde{T}])$-norm boundedness of v^j in two cases:

Let $(\tilde q,\tilde r)=(\frac{(p-1)(N+2)}{2},\frac{(p-1)(N+2)}{2})$. Case 1, if $\tilde r\ge\frac{2N}{N-2}$ and thus $(\frac{2}{1-s_c},\frac{2N}{N-2})\in\Lambda_{s_c}$, then

$$\begin{aligned}\|v^j(t)\|_{S(\dot H^{s_c};[0,\widetilde T])}&\le C(\|v^j\|_{L^\infty([0,\widetilde T];L^{\frac{2N}{N-2s_c}})}+\|v^j\|_{L^{\frac{2}{1-s_c}}([0,\widetilde T];L^{\frac{2N}{N-2}})})\\&\le C(\|v^j\|^{1-s_c}_{L^\infty([0,\widetilde T];L^2)}\|\nabla v^j\|^{s_c}_{L^\infty([0,\widetilde T];L^2)}+\widetilde T^{\frac{1-s_c}{2}}\|\nabla v^j\|_{L^\infty([0,\widetilde T];L^2)})\\&\le C(1+\widetilde T^{\frac{1-s_c}{2}})B.\end{aligned}$$

Case 2, if on the other hand $\tilde r<\frac{2N}{N-2}$. Since clearly $(\tilde q,\tilde r)\in\Lambda_{s_c}$,

$$\begin{aligned}\|v^j(t)\|_{S(\dot H^{s_c};[0,\widetilde T])}&\le C(\|v^j\|_{L^\infty([0,\widetilde T];L^{\frac{2N}{N-2s_c}})}+\|v^j\|_{L^{\frac{(p-1)(N+2)}{2}}([0,\widetilde T];L^{\frac{(p-1)(N+2)}{2}})})\\&\le C(\|v^j\|^{1-s_c}_{L^\infty([0,\widetilde T];L^2)}\|\nabla v^j\|^{s_c}_{L^\infty([0,\widetilde T];L^2)}+\widetilde T^{\frac{(p-1)(N+2)}{2}}\|\nabla v^j\|_{L^\infty([0,\widetilde T];L^2)})\\&\le C(1+\widetilde T^{\frac{(p-1)(N+2)}{2}})B.\end{aligned}$$

For fixed M , let

$$\tilde u_n(x,t)=\sum_{j=1}^{M}v^j(x-x_n^j,t-t_n^j),$$

and let

$$e_n=i\partial_t\tilde u_n+\Delta\tilde u_n+|\tilde u_n|^{p-1}\tilde u_n.$$

We claim that there exists $A=A(\widetilde T)$ (independent of M)such that for all $M>M_0$, there exists $n_0=n_0(M)$ such that for all $n>n_0$,

$$\|\tilde u_n\|_{S(\dot H^{s_c};[0,\widetilde T])}\le A.^1$$

Furthermore, we also claim that for each $M>M_0$ and $\epsilon>0$, there exists $n_1=n_1(M,\epsilon)$ such that for $n>n_1$ and for some $(q,r)\dot H^{-s_c}$ admissible,

$$\|e_n\|_{L^{q'}([0,\widetilde T];L^{r'})}\le\epsilon.$$

Both of the two claims have exactly been verified in [17](in the proof of Proposition 4.4 there) , we shall not prove them here again. Moreover, since $u_n(0)-\tilde u_n(0)=W_n^M$, there exists $M'=M'(\epsilon)$ large enough such that for each $M>M'$ there exists $n_2=n_2(M')$ such that for $n>n_2$,

$$\|e^{it\Delta}(u(0)-\tilde u(0))\|_{S(\dot H^{s_c};[0,\widetilde T])}\le\epsilon.$$

For $A=A(\widetilde T)$ in the first claim, Lemma 6.6 gives us $\epsilon_0=\epsilon_0(A)\ll 1$. We select an arbitrary $\epsilon\le\epsilon_0$ and obtain from above arguments an index $M'=M'(\epsilon)$. Now select an arbitrary $M>M'$, and set $n'=\max(n_0,n_1,n_2)$. Then by Lemma 6.6 and the above arguments, for $n>n'$, we have

$$\|u_n-\tilde{u_n}\|_{S(\dot H^{s_c};[0,T])}\le c(\widetilde T)\epsilon. \tag{6.10}$$

[1]We in fact prove both $\|\tilde u_n\|_{L^{\frac{(p-1)(N+2)}{2}}([0,\widetilde T];L^{\frac{(p-1)(N+2)}{2}})}$ and $\|\tilde u_n\|_{L^\infty([0,\widetilde T];L^{\frac{(2N)(N-2s_c)}{2}})}$ are bounded, and thus, by interpolation, for any $(q,r)\in\Lambda_{s_c}$ $(q\ge r)$, we obtain the $\|\tilde u_n\|_{L^q([0,\widetilde T];L^r)}$ bound.

In order to obtain the $\|\nabla\tilde{u}_n\|_{L^\infty([0,\widetilde{T}];L^2)}$ bound, we also have to discuss $j \geq M_2+1$. As is noted in the first paragraph of the proof, $\|v^j(t-t_n^j)\|_{S(\dot{H}^{s_c};[0,\widetilde{T}])} \to 0$ as $n \to \infty$. By Strichartz estimate we can easily get $\|\nabla v^j(t-t_n^j)\|_{L^\infty([0,\widetilde{T}];L^2)} \leq C\|\nabla v^j(-t_n^j)\|_2$. By the pairwise divergence of parameters,

$$\begin{aligned}
\|\nabla\tilde{u}_n\|^2_{L^\infty([0,\widetilde{T}];L^2)} &= \sum_{j=1}^{M_2}\|\nabla v^j(t)\|^2_{L^\infty([0,\widetilde{T}];L^2)} + \sum_{M_2+1}^{M}\|\nabla v^j(t-t_n^j)\|^2_{L^\infty([0,\widetilde{T}];L^2)} + o_n(1)\\
&\leq C\left(M_2B^2 + \sum_{M_2+1}^{M}\|\nabla NLS(-t_n^j)\psi^j\|_2^2 + o_n(1)\right)\\
&\leq C\left(M_2B^2 + \|\nabla\phi_n\|_2^2 + o_n(1)\right)\\
&\leq C\left(M_2B^2 + B^2 + o_n(1)\right).
\end{aligned}$$

Note that $\frac{2N}{N-2s_c} < p+1 < \frac{2N}{N-2}$, then for some $0<\theta<1$ and from (6.10) we have

$$\begin{aligned}
\|u_n-\tilde{u}_n\|_{L^\infty([0,\widetilde{T}];L^{p+1})} &\leq C\left(\|u_n-\tilde{u}_n\|^\theta_{L^\infty([0,\widetilde{T}];L^{\frac{2N}{N-2s_c}})}\|\nabla(u_n-\tilde{u}_n)\|^{1-\theta}_{L^\infty([0,\widetilde{T}];L^2)}\right) \qquad (6.11)\\
&\leq c(\widetilde{T})^\theta\left(M_2B^2+B^2+o_n(1)\right)^{\frac{1-\theta}{2}}\epsilon^\theta.
\end{aligned}$$

Now in the sequel we first replace the large parameter M in the notation $\tilde{u}_n$ and all other arguments above for M_1 which appears at the beginning of our proof. Then for any fixed M, we will prove (6.9) on $[0,\widetilde{T}]$. In fact, we need only to establish that, for each $t\in[0,\widetilde{T}]$,

$$\|u_n\|^{p+1}_{p+1} = \sum_{j=1}^{M}\|v^j(t-t_n^j)\|^{p+1}_{p+1} + \|W_n^M(t)\|^{p+1}_{p+1} + o_n(1). \qquad (6.12)$$

Since then by (6.3) and the energy conservation we have

$$E(u_n(t)) = \sum_{j=1}^{M}E(v^j(t-t_n^j)) + E(W_n^M(t)) + o_n(1). \qquad (6.13)$$

Thus (6.12) combined with (6.13) gives (6.9) , which completes our proof. So now what is the remainder is to establish (6.12).

We first apply the perturbation theory Lemma 6.6 to $u_n = W_n^M$ and $\tilde{u}_n = \sum_{j=M+1}^{M_1}v^j(t-t_n^j)$. For any fixed $M<M_1$, by the profile composition (6.1) and the definition of $W_n^M(t)$ and $v^j(t)$, similar to the above two claims and the arguments followed, we obtain

$$\|W_n^M(t) - \sum_{j=M+1}^{M_1}v^j(t-t_n^j)\|_{p+1} \to 0 \quad as \quad n\to\infty.$$

Then by the pairwise divergence of parameters,

$$\begin{aligned}\|u_n\|_{p+1}^{p+1} &= \|\tilde{u}_n\|_{p+1}^{p+1} + o_n(1)\\ &= \|\sum_{j=1}^{M_1} v^j(t-t_n^j)\|_{p+1}^{p+1} + o_n(1)\\ &= \sum_{j=1}^{M} \|v^j(t-t_n^j)\|_{p+1}^{p+1} + \|\sum_{j=M+1}^{M_1} v^j(t-t_n^j)\|_{p+1}^{p+1} + o_n(1)\\ &= \sum_{j=1}^{M} \|v^j(t-t_n^j)\|_{p+1}^{p+1} + \|W_n^M(t)\|_{p+1}^{p+1} + o_n(1).\end{aligned}$$

If on the other hand $M \geq M_1$, we then easily get from the selection of M_1 at the beginning of our proof that $\|W_n^M(t)\|_{p+1} = o_n(1)$ and (6.11) implies (6.12).

□

Lemma 6.8. *(Profile Reordering).Let $\phi_n(x)$ be a bounded sequence in H^1 and let $\lambda_0 > 1$. Suppose that $M(\phi_n) = M(Q)$, $E(\phi_n)/E(Q) = \omega_1\lambda_n^2 - \omega_2\lambda_n^{\frac{N(p-1)}{2}}$ with $\lambda_n \geq \lambda_0 > 1$ and $\|\nabla\phi_n\|_2/\|\nabla Q\|_2 \geq \lambda_n$ for each n. Then, for a given M, the profiles can be reordered so that there exist $1 \leq M_1 \leq M_2 \leq M$ and*
(1) For each $1 \leq j \leq M_1$, we have $t_n^j = 0$ and $v^j(t) \equiv NLS(t)\psi^j$ does not scatter as $t \to +\infty$. (We in fact assert that at least one j belongs to this category.)
(2) For each $M_1 + 1 \leq j \leq M_2$, we have $t_n^j = 0$ and $v^j(t)$ scatters as $t \to +\infty$. (There is no j in this category if $M_2 = M_1$.)
(3) For each $M_2 + 1 \leq j \leq M$ we have $|t_n^j| \to \infty$. (There is no j in this category if $M_2 = M$.)

Proof. Firstly, we prove that there exists at least one j such that t_n^j converges as $n \to \infty$. In fact,

$$\begin{aligned}\frac{\|\phi_n\|_{p+1}^{p+1}}{\|Q\|_{p+1}^{p+1}} &= -\frac{1}{\omega_2}\frac{E(\phi_n)}{E(Q)} + \frac{N(p-1)}{4}\frac{\|\nabla\phi_n\|_2^2}{\|\nabla Q\|_2^2} \\ &\geq -\frac{1}{\omega_2}\left(\omega_1\lambda_n^2 - \omega_2\lambda_n^{\frac{N(p-1)}{2}}\right) + \frac{N(p-1)}{4}\lambda_n^2 \\ &= \lambda_n^{\frac{N(p-1)}{2}} \geq \lambda_0^{\frac{N(p-1)}{2}} > 1.\end{aligned} \tag{6.14}$$

If $|t_n^j| \to \infty$, then $\|NLS(-t_n^j)\psi^j\|_{p+1} \to 0$ and (6.4) implies our claim. Now if j is such that t_n^j converges as $n \to \infty$, then we might as well assume $t_n^j = 0$.

Reorder the profiles ψ^j so that for $1 \leq j \leq M_2$, we have $t_n^j = 0$, and for $M_2 + 1 \leq j \leq M$ we have $|t_n^j| \to \infty$. What is the remainder is to show that there exists one j, $1 \leq j \leq M_2$, such that $v^j(t)$ does not scatter as $t \to +\infty$. To the contrary, if for all $1 \leq j \leq M_2$, $v^j(t)$ scatters, then we have $\lim_{t\to+\infty} \|v^j(t)\|_{p+1} = 0$. Let t_0 be sufficiently large so that for

all $1 \le j \le M_2$, we have $\|v^j(t_0)\|_{p+1}^{p+1} \le \epsilon/M_2$. The L^{p+1} orthogonality (6.12) along the NLS flow and an argument as (6.14) imply

$$\begin{aligned}\lambda_0^{\frac{N(p-1)}{2}}\|Q\|_{p+1}^{p+1} &\le \|u_n(t_0)\|_{p+1}^{p+1}\\ &= \sum_{j=1}^{M_2}\|v^j(t_0)\|_{p+1}^{p+1} + \sum_{j=M_2+1}^{M}\|v^j(t_0-t_n^j)\|_{p+1}^{p+1} + \|W_n^M(t_0)\|_{p+1}^{p+1} + o_n(1).\end{aligned}$$

We know from Proposition 6.5 that,as $n \to +\infty$, $\sum_{j=M_2+1}^{M}\|v^j(t_0-t_n^j)\|_{p+1}^{p+1} \to 0$, and thus we have

$$\lambda_0^{\frac{N(p-1)}{2}}\|Q\|_{p+1}^{p+1} \le \epsilon + \|W_n^M(t_0)\|_{p+1}^{p+1} + o_n(1).$$

This gives a contradiction since $W_n^M(t)$ is a scattering solution.

□

7. Inductive Argument and Existence of a Critical Solution

We now begin to prove Theorem 1.1. Note from Remark 1.2 that we have reduced Theorem 1.1 to the case $P(u) = 0$, thus we first give some definitions :

Definition 7.1. Let $\lambda > 1$. We say that $\exists GB(\lambda, \sigma)$ holds if there exists a solution $u(t)$ to (1.1) such that

$$P(u) = 0, \quad M(u) = M(Q), \quad \frac{E(u)}{E(Q)} = \omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}$$

and

$$\lambda \le \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2} \le \sigma \quad for\ all\ \ t \ge 0.$$

$\exists GB(\lambda, \sigma)$ means that there exist solutions with energy $\omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}$ globally bounded by σ. Thus by Proposition 5.1 , $\exists GB(\lambda, \lambda(1+\rho_0(\lambda_0)))$ is false for all $\lambda \ge \lambda_0 > 1$.

The statement $\exists GB(\lambda, \sigma)$ is false is equivalent to say that for every solution $u(t)$ to (1.1) with $M(u) = M(Q)$ and $E(u)/E(Q) = \omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}$ such that $\|\nabla u(t)\|_2/\|\nabla Q\|_2 \ge \lambda$ for all t, there must exists a time $t_0 \ge 0$ such that $\|\nabla u(t_0)\|_2/\|\nabla Q\|_2 \ge \sigma$. By resetting the initial time, we can find a sequence $t_n \to \infty$ such that $\|\nabla u(t_n)\|_2/\|\nabla Q\|_2 \ge \sigma$ for all n.

Note that if $\lambda \le \sigma_1 \le \sigma_2$, then $\exists GB(\lambda, \sigma_2)$ is false implies $\exists GB(\lambda, \sigma_1)$ is false. We will induct on the statement and define a threshold.

Definition 7.2. (The Critical Threshold.) Fix $\lambda_0 > 1$. Let $\sigma_c = \sigma_c(\lambda_0)$ be the supremum of all $\sigma > \lambda_0$ such that $\exists GB(\lambda, \sigma)$ is false for all λ such that $\lambda_0 \le \lambda \le \sigma$.

Proposition 5.1 implies that $\sigma_c(\lambda_0) > \lambda_0$. Let $u(t)$ be any solution to (1.1) with $P(u) = 0$, $M(u) = M(Q)$, $E(u)/E(Q) \le \omega_1\lambda_0^2 - \omega_2\lambda_0^{\frac{N(p-1)}{2}}$ and $\|\nabla u(0)\|_2/\|\nabla Q\|_2 > 1$. If $\lambda_0 > 1$ and $\sigma_c = \infty$, we claim that there exists a sequence of times t_n such that $\|\nabla u(t_n)\|_2 \to \infty$. In fact, if not, and let $\lambda \ge \lambda_0$ be such that $E(u)/E(Q) = \omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}$. Since

there is no sequence t_n such that $\|\nabla u(t_n)\|_2 \to \infty$, there must exists $\sigma < \infty$ such that $\lambda \leq \|\nabla u(t)\|_2/\|\nabla Q\|_2 \leq \sigma$ for all $t \geq 0$, which means that $\exists GB(\lambda, \sigma)$ holds true. Thus $\sigma_c \leq \sigma < \infty$ and we get a contradiction.

In view of the above claim, if we can prove that for every $\lambda_0 > 1$ then $\sigma_c(\lambda_0) = \infty$, we then have in fact proved our Theorem 1.1. Thus, in the sequel, we shall carry it out by contradiction; more precisely , fix $\lambda_0 > 1$ and assume $\sigma_c < \infty$, we shall work toward a absurdity. (It, of course, suffices to do this for λ_0 close to 1, and so we might as well assume that $\lambda_0 < (\frac{\omega_1}{\omega_2})^{\frac{2}{N(p-1)-4}}$, which will be convenient in the sequel.) For that purpose, we need first to obtain the existence of a critical solution:

Lemma 7.3. $\sigma_c(\lambda_0) < \infty$. *Then there exist initial data $u_{c,0}$ and $\lambda_c \in [\lambda_0, \sigma_c(\lambda_0)]$ such that $u_c(t) \equiv NLS(t)u_{c,0}$ is global, $P(u_c) = 0$, $M(u_c) = M(Q)$, $E(u_c)/E(Q) = \omega_1\lambda_c^2 - \omega_2\lambda_c^{\frac{N(p-1)}{2}}$, and*

$$\lambda_c \leq \frac{\|\nabla u_c(t)\|_2}{\|\nabla Q\|_2} \leq \sigma_c \quad for\ all \quad t \geq 0.$$

We call u_c a critical solution since by definition of σ_c we have that for all $\sigma < \sigma_c$ and all $\lambda_0 \leq \lambda \leq \sigma$, $\exists GB(\lambda, \sigma)$ is false, i.e., there are no solutions $u(t)$ for which

$$P(u) = 0, \qquad M(u) = M(Q), \qquad \frac{E(u)}{E(Q)} = \omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}$$

and

$$\lambda \leq \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2} \leq \sigma \quad for\ all \quad t \geq 0.$$

Proof. By definition of σ_c, there exist sequence λ_n and σ_n such that $\lambda_0 \leq \lambda_n \leq \sigma_n$ and $\sigma_n \downarrow \sigma_c$ for which $\exists GB(\lambda_n, \sigma_n)$ holds. This means that there exists $u_{n,0}$ such that $u_n(t) \equiv NLS(t)u_{n,0}$ is global with $P(u_n) = 0$, $M(u_n) = M(Q)$, $E(u_n)/E(Q) = \omega_1\lambda_n^2 - \omega_2\lambda_n^{\frac{N(p-1)}{2}}$, and

$$\lambda_n \leq \frac{\|\nabla u_n(t)\|_2}{\|\nabla Q\|_2} \leq \sigma_n \quad for\ all \quad t \geq 0.$$

The boundedness of λ_n make us pass to a subsequence such that λ_n converges with a limit $\lambda' \in [\lambda_0, \sigma_c]$.

According to Lemma 6.8 where we take $\phi_n = u_{n,0}$, for $M_1 + 1 \leq j \leq M_2$, $v^j(t) \equiv NLS(t)\psi^j$ scatter as $t \to +\infty$ and combined with Proposition 6.5, for $M_2 + 1 \leq j \leq M$, v^j also scatter in one or the other time direction. Thus by the scattering theory, for $M_1 + 1 \leq j \leq M$, we have $E(v_j) = E(\psi_j) \geq 0$ and then by (6.3)

$$\sum_{j=1}^{M_1} E(\psi^j) \leq E(\phi_n) + o_n(1).$$

Thus there exists at least one $1 \leq j \leq M_1$ with

$$E(\psi^j) \leq \max(\lim_n E(\phi_n), 0),$$

which, without loss of generality, we might as well take $j = 1$. Since, by the profile composition, also $M(\psi^1) \le \lim_n M(\phi_n) = M(Q)$, we then have

$$\frac{M^{\frac{1-s_c}{s_c}}(\psi^1)E(\psi^1)}{M^{\frac{1-s_c}{s_c}}(Q)E(Q)} \le \max\left(\lim_n \frac{E(\phi_n)}{E(Q)}, 0\right).$$

Thus, there exist $\tilde\lambda \ge \lambda_0$ [2]such that

$$\frac{M^{\frac{1-s_c}{s_c}}(\psi^1)E(\psi^1)}{M^{\frac{1-s_c}{s_c}}(Q)E(Q)} = \omega_1\tilde\lambda^2 - \omega_2\tilde\lambda^{\frac{N(p-1)}{2}}.$$

Note that by Lemma 6.8, v^1 does not scatter, so it follows from Theorem 2.5 that $\|\psi^1\|_2^{\frac{1-s_c}{s_c}}\|\nabla\psi^1\|_2 < \|Q\|_2^{\frac{1-s_c}{s_c}}\|\nabla Q\|_2$ cannot hold. Then by Proposition 2.2, we must have $\|\psi^1\|_2^{\frac{1-s_c}{s_c}}\|\nabla\psi^1\|_2 \ge \tilde\lambda\|Q\|_2^{\frac{1-s_c}{s_c}}\|\nabla Q\|_2$.

Now if $\tilde\lambda > \sigma_c$ and recall that $t_n^1 = 0$, then for all t we know that

$$\tilde\lambda^2 \le \frac{\|v^1(t)\|_2^{2\frac{1-s_c}{s_c}}\|\nabla v^1(t)\|_2^2}{\|Q\|_2^{2\frac{1-s_c}{s_c}}\|\nabla Q\|_2^2} \le \frac{\|\nabla v^1(t)\|_2^2}{\|\nabla Q\|_2^2} \le \frac{\sum_{j=1}^M \|\nabla v^j(t-t_n^j)\|_2^2 + \|\nabla W_n^M(t)\|_2^2}{\|\nabla Q\|_2^2}. \quad (7.1)$$

Taking $t = 0$ for example, Lemma 6.7 implies that

$$\tilde\lambda^2 \le \frac{\sum_{j=1}^M \|\nabla v^j(-t_n^j)\|_2^2 + \|\nabla W_n^M\|_2^2}{\|\nabla Q\|_2^2} \le \frac{\|\nabla u_n(0)\|_2^2}{\|\nabla Q\|_2^2} + o_n(1) \le \sigma_c^2 + o_n(1)$$

which contradicts the assumption $\tilde\lambda > \sigma_c$. Hence we must have $\tilde\lambda \le \sigma_c$.

Now if $\tilde\lambda < \sigma_c$, we know from the definition of σ_c that $\exists GB(\tilde\lambda, \sigma_c - \delta)$ is false for any $\delta > 0$ sufficiently small, and then there exists a nondecreasing sequence t_k of times such that

$$\lim_k \frac{\|v^1(t_k)\|_2^{\frac{1-s_c}{s_c}}\|\nabla v^1(t_k)\|_2}{\|Q\|_2^{\frac{1-s_c}{s_c}}\|\nabla Q\|_2} \ge \sigma_c.$$

Note that $t_n^1 = 0$, then

$$\begin{aligned}\sigma_c^2 - o_k(1) &\le \frac{\|v^1(t_k)\|_2^{2\frac{1-s_c}{s_c}}\|\nabla v^1(t_k)\|_2^2}{\|Q\|_2^{2\frac{1-s_c}{s_c}}\|\nabla Q\|_2^2} \le \frac{\|\nabla v^1(t_k)\|_2^2}{\|\nabla Q\|_2^2} \quad (7.2)\\ &\le \frac{\sum_{j=1}^M \|\nabla v^j(t_k - t_n^j)\|_2^2 + \|\nabla W_n^M(t_k)\|_2^2}{\|\nabla Q\|_2^2}\\ &\le \frac{\|\nabla u_n(t)\|_2^2}{\|\nabla Q\|_2^2} + o_n(1)\\ &\le \sigma_c^2 + o_n(1),\end{aligned}$$

[2]If $\lim_n E(\phi_n) \ge 0$, we have $\tilde\lambda \ge \lambda' \ge \lambda_0$; while in the case $\lim_n E(\phi_n) < 0$, we will have $\tilde\lambda \ge (\frac{\omega_1}{\omega_2})^{\frac{2}{N(p-1)-4}} > \lambda_0$ though we might not have $\tilde\lambda \ge \lambda'$.

where by Lemma 6.7 we take $n = n(k)$ large. Sending $k \to \infty$ and hence $n(k) \to \infty$, we conclude that all inequalities must be equalities. Thus we conclude that $W_n^M(t_k) \to 0$ in H^1, $M(v^1) = M(Q)$ and $v^j \equiv 0$ for all $j \geq 2$. Thus easily $P(v^1) = P(u_n) = 0$. On the other hand if $\tilde{\lambda} = \sigma_c$, we need not the inductive hypothesis but, similar to (7.1), obtain

$$\sigma_c^2 \leq \frac{\sum_{j=1}^M \|\nabla v^j(-t_n^j)\|_2^2 + \|\nabla W_n^M\|_2^2}{\|\nabla Q\|_2^2} \leq \frac{\|\nabla u_n(0)\|_2^2}{\|\nabla Q\|_2^2} + o_n(1) \leq \sigma_c^2 + o_n(1),$$

and then again, we conclude that $W_n^M \to 0$ in H^1, $M(v^1) = M(Q)$ and $v^j \equiv 0$ for all $j \geq 2$. Moreover, by Lemma 6.7, for all t

$$\frac{\|\nabla v^1(t)\|_2^2}{\|\nabla Q\|_2^2} \leq \lim_n \frac{\|\nabla u_n(t)\|_2^2}{\|\nabla Q\|_2^2} \leq \sigma_c^2.$$

Hence, we take $u_{c,0} = v^1(0) = \psi^1$ and $\lambda_c = \tilde{\lambda}$ to complete our proof.

□

8. Concentration of Critical Solutions and Proof of Theorem 1.1

In this section, we will finally prove Theorem 1.1 by virtue of the precompactness of the flow of the critical solution. To simplify notation, we take $u(t) = u_c(t)$ in the sequel.

Lemma 8.1. *There exists a path $x(t)$ in $\mathbb{R}^N$ such that*

$$K \equiv \{u(t, \cdot - x(t)) | t \geq 0\} \subset H^1$$

is precompact in H^1.

Proof. As is showed in [2] , it suffices to prove that for each sequence of times $t_n \to \infty$, there exists a sequence x_n such that, by passing to a subsequence, $u(t_n, \cdot - x_n)$ converges in H^1.

Taking $\phi_n = u(t_n)$ in Lemma 6.8 and by definition of $u(t) = u_c(t)$, similar to the proof of Lemma 7.3, we obtain that there exists at least one $1 \leq j \leq M_1$ with

$$E(\psi^j) \leq \max(\lim_n E(\phi_n), 0).$$

Without loss of generality, we can take $j = 1$. Since, also $M(\psi^1) \leq \lim_n M(\phi_n) = M(Q)$, there exist $\tilde{\lambda} \geq \lambda_0$ such that

$$\frac{M^{\frac{1-s_c}{s_c}}(\psi^1)E(\psi^1)}{M^{\frac{1-s_c}{s_c}}(Q)E(Q)} = \omega_1\tilde{\lambda}^2 - \omega_2\tilde{\lambda}^{\frac{N(p-1)}{2}}.$$

Note that by Lemma 6.8, v^1 does not scatter, so we must have $\|\psi^1\|_2\|\nabla\psi^1\|_2 \geq \tilde{\lambda}\|Q\|_2\|\nabla Q\|_2$. Then by the same way as in the proof of Lemma 7.3 , we get that $W_n^M(t_k) \to 0$ in H^1 and $v^j \equiv 0$ for all $j \geq 2$. Since we know that $W_n^M(t)$ is a scattering solution , this implies that

$$W_n^M(0) = W_n^M \to 0 \quad in \quad H^1. \tag{8.1}$$

Consequently, we have

$$u(t_n) = NLS(-t_n^1)\psi^1(x - x_n^1) + W_n^M(x).$$

Note that by Lemma 6.8, $t_n^1 = 0$, and thus

$$u(t_n, x + x_n^1) = \psi^1(x) + W_n^M(x + x_n^1).$$

This equality and (8.1) imply our conclusion. □

Using the uniform-in-time H^1 concentration of $u(t) = u_c(t)$ and by changing of variables, we can easily get

Corollary 8.2. *For each $\epsilon > 0$, there exists $R > 0$ such that for all t,*

$$\|u(t, \cdot - x(t))\|_{H^1(|x| \geq R)} \leq \epsilon.$$

With the localization property of u_c, we show, similar to [7], that u_c must blow up in finite time using the same method as that in the proof of Proposition 3.2, which contradicts the boundedness of u_c in H^1. Hence, u_c cannot exist and $\sigma_c = \infty$. As is argued in section 7, this indeed completes the proof of Theorem 1.1.

Appendix A. Nonzero Momentum

Suppose that the solution $u(x,t)$ with $M(u) = M(Q)$, $P(u) \neq 0$. Applying Galilean transform to $u(x,t)$, we obtain a new solution $\tilde{u}(x,t)$:

$$\tilde{u}(x,t) = e^{ix\cdot\xi_0} e^{-it|\xi_0|^2} u(x - 2\xi_0 t, t).$$

Take $\xi_0 = -\frac{P(u)}{M(u)}$ and we get

$$P(\tilde{u}) = 0, \quad M(\tilde{u}) = M(u) = M(Q), \quad \|\nabla\tilde{u}\|_2^2 = \|\nabla u\|_2^2 - \frac{P(u)^2}{M(u)}$$

and

$$E(\tilde{u}) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{M(u)}{2}\left(\xi_0 + \frac{P(u)}{M(u)}\right)^2 - \frac{P(u)^2}{2M(u)} = E(u) - \frac{1}{2}\frac{P(u)^2}{M(u)}.$$

Thus this choice of ξ_0 make $E(\tilde{u})$ attain its lowest value under any choice of $\xi_0 \in \mathbb{R}^N$. And as is stated in [7], $E(\tilde{u}) < E(u) < E(Q)$ implies that we should always implement this transformation to maximize the applicability of Proposition 2.2.

Now what we should do is to show that if the dichotomy of Proposition 2.2 was already valid for u, then the selection of case (1) versus (2) in Proposition 2.2 is preserved under the Galilean transformation.

Suppose $M(u) = M(Q)$, $E(u) < E(Q)$ and $P(u) \neq 0$. Define $\tilde{u}(x,t)$ as above. Let λ_-, λ be defined in terms of $E(u)$ by (2.6) and $\eta(t)$ in terms of $u(t)$ by (2.4). Let$\tilde{\lambda}_-, \tilde{\lambda}$ and $\tilde{\eta}(t)$ be the same quantities associated to $\tilde{u}$.

Firstly, suppose that case (1) of Proposition 2.2 holds for u, which in particular implies that $\eta(t) < 1$ for all t. But clearly $\tilde{\eta}(t) < \eta(t) < 1$, thus, case (1) of Proposition 2.2 holds for $\tilde{u}$ also.

Now conversely, suppose that case (1) of Proposition 2.2 holds for $\tilde{u}$, then $\tilde{\eta}(t)^2 \leq \tilde{\lambda}_-^2$ for all t. We claim that

$$\eta(t)^2 = \frac{\|\nabla u\|_2^2}{\|\nabla Q\|_2^2} = \tilde{\eta}(t)^2 + \frac{P(u)^2}{2M(u)\|\nabla Q\|_2^2} = \tilde{\eta}(t)^2 + \frac{P(u)^2}{2\omega_1 M(u)E(Q)} \leq \lambda_-^2.$$

Indeed, this reduced to an algebraic problem now. Denote $\alpha = \frac{E(u)}{E(Q)}$ and $\beta = \frac{P(u)^2}{M(u)E(Q)}$. Then $\tilde{\lambda}_-$ is the smaller root of the equation:

$$\omega_1 \tilde{\lambda}_-^2 - \omega_2 \tilde{\lambda}_-^{\frac{N(p-1)}{2}} = \frac{E(\tilde{u})}{E(Q)} = \frac{E(u)}{E(Q)} - \frac{P(u)^2}{2M(u)E(Q)} = \alpha - \frac{\beta}{2},$$

while λ_- is the smaller root of

$$\omega_1 \lambda_-^2 - \omega_2 \lambda_-^{\frac{N(p-1)}{2}} = \frac{E(u)}{E(Q)} = \alpha.$$

Let the function $f(x) = \omega_1 x - \omega_2 x^{\frac{N(p-1)}{4}}$. Observe that the above claim follows if we could prove that $f(\tilde{\lambda}_-^2 + \frac{\beta}{2\omega_1}) \leq f(\lambda_-^2)$. Equivalently, it suffices to show $f(\tilde{\lambda}_-^2 + \frac{\beta}{2\omega_1}) \leq f(\tilde{\lambda}_-^2) + \frac{\beta}{2}$, or

$$f(\tilde{\lambda}_-^2 + \frac{\beta}{2\omega_1}) - f(\tilde{\lambda}_-^2) \leq \frac{\beta}{2}. \tag{A.1}$$

The left hand side of (A.1) is $\frac{\beta}{2} - \omega_2 \left((\tilde{\lambda}_-^2 + \frac{\beta}{2\omega_1})^{\frac{N(p-1)}{4}} - (\tilde{\lambda}_-^2)^{\frac{N(p-1)}{4}} \right)$ which is certainly no larger than $\frac{\beta}{2}$ since $p - 1 > \frac{4}{N}$, and we conclude our claim.

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NONSCATTERING SOLUTIONS TO THE L^2 -SUPERCRITICAL NLS EQUATIONS

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Abstract. We investigate the nonlinear Schrödinger equation $iu_t + \Delta u + |u|^{p-1}u = 0$ with $1+\frac{4}{N} < p < 1+\frac{4}{N-2}$ (when $N = 1, 2$, $1+\frac{4}{N} < p < \infty$) in energy space H^1 and study the divergent property of infinite-variance and nonradial solutions. If $M(u)^{\frac{1-s_c}{s_c}} E(u) < M(Q)^{\frac{1-s_c}{s_c}} E(Q)$ and $\|u_0\|_2^{\frac{1-s_c}{s_c}} \|\nabla u_0\|_2 > \|Q\|_2^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2$, then either $u(t)$ blows up in finite forward time, or $u(t)$ exists globally for positive time and there exists a time sequence $t_n \to +\infty$ such that $\|\nabla u(t_n)\|_2 \to +\infty$. Here Q is the ground state solution of $-Q+\Delta Q+|Q|^{p-1}Q = 0$. A similar result holds for negative time. This extend the result of the 3D cubic Schrödinger equation in [8] to the general mass-supercritical and energy-subcritical case .

MSC: 35Q55, 35A15, 35B30.

Keywords: Nonlinear Schrödinger equation; Blow-up solution; Infinite variance; Mass-supercritical; Energy-subcritical

1. Introduction

We consider the following Cauchy problem of a nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u + |u|^{p-1}u = 0, \quad (x,t) \in R^N \times R, \\ \qquad u(x,0) = u_0(x) \in H^1(R^N). \end{cases} \tag{1.1}$$

It is well known from Ginibre and Velo [4] that, equation (1.1) is locally well-posed in H^1. That is for $u_0 \in H^1$, there exist $0 < T \leq \infty$ and a unique solution $u(t) \in C([0,T); H^1)$ to (1.1). When $T = \infty$, we say that the solution is positively global; while on the other hand, we have $\lim_{t\uparrow T} \|\nabla u(t)\|_2 \to \infty$ and call that this solution blows up in finite positive time. Solutions of (1.1) admits the following conservation laws in energy space H^1 :

$$L^2 - norm: \quad M(u)(t) \equiv \int |u(x,t)|^2 dx = M(u_0);$$

$$Energy: \quad E(u)(t) \equiv \frac{1}{2}\int |\nabla u(x,t)|^2 dx - \frac{1}{p+1}\int |u(x,t)|^{p+1} dx = E(u_0);$$

$$Momentum: \quad P(u)(t) \equiv Im \int \overline{u}(x,t)\nabla u(x,t) dx = P(u_0).$$

Note that equation (1.1) is invariant under the scaling $u(x,t) \to \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t)$ which also leaves the homogeneous Sobolev norm $\dot{H}^{s_c}$ invariant with $s_c = \frac{N}{2} - \frac{2}{p-1}$. It is classical from the conservation of the energy and the L^2 norm that for $s_c < 0$, the equation is subcritical and all H^1 solutions are global and H^1 bounded. The smallest power for which blow up may occur is $p = 1 + \frac{4}{N}$ which is referred to as the L^2 critical case corresponding to $s_c = 0$ [5] [13]. The case $0 < s_c < 1$ is called the L^2 supercritical and H^1 subcritical or the Mass-supercritical and Energy-subcritical case. In fact, we are concerning in this paper with the case $0 < s_c < 1$.

For the 3D cubic nonlinear Schrödinger equation with $s_c = \frac{1}{2}$ and $p = 3$, there have been several results on either scattering or blow-up solutions. In Holmer and Roudenko [7], the authors proved that if $u_0 \in H^1$ is radial, $M(u)E(u) < M(Q)E(Q)$ and $\|\nabla u_0\|_2 \|u_0\|_2 < \|\nabla Q\|_2 \|Q\|_2$, then the solution $u(t)$ is globally well-posed and scattering; They further showed that if $M(u)E(u) < M(Q)E(Q)$ and $\|\nabla u_0\|_2 \|u_0\|_2 > \|\nabla Q\|_2 \|Q\|_2$, then the solution blows up in finite time, provided that either the initial data has finite variance or is radial. The radial case is an extension of a result of Ogawa and Tsutsumi [15] who proved the case $E(u) < 0$. Then in [3], also for the 3D cubic nonlinear Schrödinger equation, the authors extended the scattering results on radial H^1 solutions to the nonradial case . The technique employed is parallel to that employed by Kenig-Merle [10] in their study of the energy-critical NLS. For $0 < s_c < 1$, [18] have extended the scattering results to the general L^2 supercritical and H^1 subcritical case.

Then in Holmer and Roudenko [8], the authors further studied the blow-up theory for the 3D cubic nonlinear Schrödinger equation , which dropped the additional hypothesis of finite variance and radiality. More precisely, they proved that if $M(u)E(u) < M(Q)E(Q)$ and $\|\nabla u_0\|_2 \|u_0\|_2 > \|\nabla Q\|_2 \|Q\|_2$, then either $u(t)$ blows up in finite positive time, or $u(t)$ exists globally for all positive time and there exists a time sequence $t_n \to +\infty$ such that $\|\nabla u(t_n)\|_2 \to \infty$, with similar results holding for negative time.

In this paper, we extend the above results to the general L^2 supercritical and H^1 subcritical case, and obtain the following conclusion:

Theorem 1.1. *Suppose* $u_0 \in H^1$, $M(u)^{\frac{1-s_c}{s_c}} E(u) < M(Q)^{\frac{1-s_c}{s_c}} E(Q)$ *and* $\|\nabla u_0\|_2 \|u_0\|_2^{\frac{1-s_c}{s_c}} > \|\nabla Q\|_2 \|Q\|_2^{\frac{1-s_c}{s_c}}$. *Then either* $u(t)$ *blows up in finite forward time, or* $u(t)$ *is forward global and there exists a time sequence* $t_n \to \infty$ *such that* $\|\nabla u(t_n)\|_2 \to \infty$. *A similar statement holds for negative time.*

Different from a similar result obtained by Glangeta and Merle [6] for the case $E(u) < 0$, our proof is by means of the profile decomposition introduced by Keraani [12], nonlinear pertubation theory based on the Strichartz estimate [2] [11], and the rigidity theorems based upon the localized virial identity. Though with the same idea as in [8], we still have to reestablish the tools mentioned above, such as the profile decomposition, to conquer the difficulties our general case should bring .

Remark 1.2. Via the Galilean transform and momentum conservation, in this paper, we will always assume that $P(u) = 0$, and put further standard details in the Appendix. That is to say we need only show Theorem 1.1 under the condition $P(u) = 0$.

In this paper, we denote the Sobolev space $H^1(\mathbb{R}^N)$ as H^1 for short, and the L^p norm $\|\cdot\|_p$. Also for convenience, we will use the notation C, except for some specifications, standing for the variant absolute constants.

After this paper was finished, we learnt that the authors in [1] has obtained the same result presented in this paper. However, the proof here is different from that in [1]. We hope our method can be helpful for other related problem.

2. Preliminaries

In this section, we will review some basic facts about the ground state and give a dichotomy result.

Weinstein in [17] proved that the sharp constant C_{GN} of Gagliardo-Nirenberg inequality for $0 < s_c < 1$

$$\|u\|^{p+1}_{L^{p+1}(R^N)} \leq C_{GN} \|\nabla u\|^{\frac{N(p-1)}{2}}_{L^2(R^N)} \|u\|^{2-\frac{(N-2)(p-1)}{2}}_{L^2(R^N)} \tag{2.1}$$

is achieved by $u = Q$, where Q is the ground state of

$$-(1-s_c)Q + \Delta Q + |Q|^{p-1}Q = 0.$$

Using Pohozhaev identities we can get the following identities without difficulty:

$$\|Q\|_2^2 = \frac{2}{N}\|\nabla Q\|_2^2,$$

$$\|Q\|^{p+1}_{p+1} = \frac{2(p+1)}{N(p-1)}\|\nabla Q\|_2^2 = \frac{(p+1)}{(p-1)}\|Q\|_2^2,$$

$$E(Q) = \frac{N(p-1)-4}{2N(p-1)}\|\nabla Q\|_2^2 = \frac{N(p-1)-4}{4(p-1)}\|Q\|_2^2 = \frac{N(p-1)-4}{4(p+1)}\|Q\|^{p+1}_{p+1}, \tag{2.2}$$

and C_{GN} can be expressed by

$$C_{GN} = \frac{\|Q\|^{p+1}_{p+1}}{\|\nabla Q\|_2^{\frac{N(p-1)}{2}} \|Q\|_2^{2-\frac{(N-2)(p-1)}{2}}}. \tag{2.3}$$

Note that the Sobolev $\dot{H}^{s_c}$ norm and the equation (1.1) are invariant under the scaling $u(x,t) \mapsto u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t)$. Other scaling invariant quantities are $\|\nabla u\|_2 \|u\|_2^{\frac{1-s_c}{s_c}}$ and $E(u)M(u)^{\frac{1-s_c}{s_c}}$.

Let

$$\eta(t) = \frac{\|\nabla u\|_2 \|u\|_2^{\frac{1-s_c}{s_c}}}{\|\nabla Q\|_2 \|Q\|_2^{\frac{1-s_c}{s_c}}}. \tag{2.4}$$

In order to study the relationship between $\eta(t)$ and $\frac{E(u)M(u)^{\frac{1-s_c}{s_c}}}{E(Q)M(Q)^{\frac{1-s_c}{s_c}}}$, we might as well assume $\|u\|_2 = \|Q\|_2$ by scaling. Denote $\omega_1 = \frac{N(p-1)}{N(p-1)-4}$ and $\omega_2 = \frac{4}{N(p-1)-4}$. Then by (2.1)-(2.3) we have

$$\begin{aligned}
&2\omega_1\frac{\|\nabla u\|_2^2\|u\|_2^{\frac{2-2s_c}{s_c}}}{\|\nabla Q\|_2^2\|Q\|_2^{\frac{2-2s_c}{s_c}}} \geq \frac{E(u)M(u)^{\frac{1-s_c}{s_c}}}{E(Q)M(Q)^{\frac{1-s_c}{s_c}}} = \frac{E(u)}{E(Q)}\\
&= \omega_1\frac{\|\nabla u\|_2^2}{\|\nabla Q\|_2^2} - \frac{2\omega_1}{p+1}\frac{\|u\|_{p+1}^{p+1}}{\|Q\|_{p+1}^{p+1}}\\
&\geq \omega_1\frac{\|\nabla u\|_2^2}{\|\nabla Q\|_2^2} - \frac{2\omega_1}{p+1}\frac{C_{GN}\|\nabla u\|_2^{\frac{N(p-1)}{2}}\|u\|_2^{2-\frac{(N-2)(p-1)}{2}}}{\|Q\|_2^2}\\
&= \omega_1\eta(t)^2 - \frac{2\omega_1}{p+1}\frac{C_{GN}\|Q\|_2^{2-\frac{(N-2)(p-1)}{2}}}{\|\nabla Q\|_2^{2-\frac{(N-2)(p-1)}{2}}}\frac{\|\nabla u\|_2^{\frac{N(p-1)}{2}}}{\|\nabla Q\|_2^{\frac{N(p-1)}{2}}}\\
&= \omega_1\eta(t)^2 - \frac{4\omega_1}{N(p-1)}\eta(t)^{\frac{N(p-1)}{2}} = \omega_1\eta(t)^2 - \omega_2\eta(t)^{\frac{N(p-1)}{2}}.
\end{aligned}$$

That is

$$2\omega_1\eta(t)^2 \geq \frac{E(u)M(u)^{\frac{1-s_c}{s_c}}}{E(Q)M(Q)^{\frac{1-s_c}{s_c}}} \geq \omega_1\eta(t)^2 - \omega_2\eta(t)^{\frac{N(p-1)}{2}}. \tag{2.5}$$

Note that $\frac{\omega_1}{\omega_2} > 1$ as $\frac{4}{N} < p-1 < \frac{4}{N-2}$. Thus it is not difficult to observe that if $0 \leq M(u)^{\frac{1-s_c}{s_c}}E(u)/M(Q)^{\frac{1-s_c}{s_c}}E(Q) < 1$, then there exist two solutions $0 \leq \lambda_- < 1 < \lambda$ to the following equation of λ

$$\frac{E(u)M(u)^{\frac{1-s_c}{s_c}}}{E(Q)M(Q)^{\frac{1-s_c}{s_c}}} = \omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}. \tag{2.6}$$

By the H^1 local theory [2] , there exist $-\infty \leq T_- < 0 < T_+ \leq +\infty$ such that (T_-, T_+) is the maximal time interval of existence for $u(t)$ solving (1.1) , and if $T_+ < +\infty$ then

$$\|\nabla u(t)\|_2 \geq \frac{C}{(T_+-t)^{\frac{1}{p-1}-\frac{N-2}{4}}} \quad as\ t \uparrow T_+,$$

and a similar argument holds if $-\infty < T_-$. Moreover, as a consequence of the continuity of the flow $u(t)$, we have the following dichotomy proposition :

Proposition 2.1. *(Global versus blow-up dichotomy) Let $u_0 \in H^1(R^N)$, and let $I = (T_-, T_+)$ be the maximal time interval of existence of $u(t)$ solving* (1.1). *Suppose that*

$$M(u)^{\frac{1-s_c}{s_c}}E(u) < M(Q)^{\frac{1-s_c}{s_c}}E(Q). \tag{2.7}$$

If (2.7) *holds and*

$$\|u_0\|_2^{\frac{1-s_c}{s_c}}\|\nabla u_0\|_2 < \|Q\|_2^{\frac{1-s_c}{s_c}}\|\nabla Q\|_2, \tag{2.8}$$

then $I = (-\infty, +\infty)$, i.e., the solution exists globally in time, and for all time $t \in \mathbb{R}$,

$$\|u(t)\|_2^{\frac{1-s_c}{s_c}}\|\nabla u(t)\|_2 < \|Q\|_2^{\frac{1-s_c}{s_c}}\|\nabla Q\|_2. \tag{2.9}$$

If (2.7) *holds and*

$$\|u_0\|_2^{\frac{1-s_c}{s_c}}\|\nabla u_0\|_2 > \|Q\|_2^{\frac{1-s_c}{s_c}}\|\nabla Q\|_2, \tag{2.10}$$

then for $t \in I$,

$$\|u(t)\|_2^{\frac{1-s_c}{s_c}}\|\nabla u(t)\|_2 > \|Q\|_2^{\frac{1-s_c}{s_c}}\|\nabla Q\|_2. \tag{2.11}$$

Proof. Multiplying the formula of energy by $M(u)^{\frac{1}{s_c}-1}$ and using the Gagliardo-Nirenberg inequality we have

$$\begin{aligned} E(u)M(u)^{\frac{1}{s_c}-1} &= \frac{1}{2}\|\nabla u\|_{L^2}^2\|u\|_{L^2}^{\frac{2}{s_c}-2} - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}\|u\|_{L^2}^{\frac{2}{s_c}-2} \\ &\geq \frac{1}{2}(\|\nabla u\|_2\|u\|_2^{\frac{1-s_c}{s_c}})^2 - \frac{1}{p+1}C_{GN}(\|\nabla u\|_2\|u\|_2^{\frac{1-s_c}{s_c}})^{\frac{N(p-1)}{2}}. \end{aligned}$$

Define $f(x) = \frac{1}{2}x^2 - \frac{1}{p+1}C_{GN}x^{\frac{N(p-1)}{2}}$. Since $N(p-1) \geq 4$, then $f'(x) = x(1 - C_{GN}\frac{N(p-1)}{2(p+1)}x^{\frac{N(p-1)-4}{2}})$, and $f'(x) = 0$ when $x_0 = 0$ and $x_1 = \|\nabla Q\|_2\|Q\|_2^{\frac{1-s_c}{s_c}}$. Note that $f(0) = 0$ and $f(x_1) = E(u)M(u)^{\frac{1}{s_c}-1}$, thus the graph of f has two extrema: a local minimum at x_0 and a local maximum at x_1. The condition (2.7) implies that $E(u_0)M(u_0)^{\frac{1}{s_c}-1} < f(x_1)$. Combining with energy conservation, we have

$$f(\|\nabla u\|_2\|u\|_2^{\frac{1-s_c}{s_c}}) \leq E(u)M(u_0)^{\frac{1}{s_c}-1} = E(u)M(u)^{\frac{1}{s_c}-1} < f(x_1). \tag{2.12}$$

If initially $\|\nabla u_0\|_2\|u_0\|_2^{\frac{1-s_c}{s_c}} < x_1$, i.e., the condition (2.8) holds, then by (2.12) and the continuity of $\|\nabla u(t)\|_2$ in t, we have $\|\nabla u(t)\|_2\|u(t)\|_2^{\frac{1-s_c}{s_c}} < x_1$ for all time $t \in I$. In particular, the H^1 norm of the solution is bounded, which implies the global existence and (2.9) in this case.

If initially $\|\nabla u_0\|_2\|u_0\|_2^{\frac{1-s_c}{s_c}} > x_1$, i.e., the condition (2.10) holds, then by (2.12) and the continuity of $\|\nabla u(t)\|_2$ in t, we have $\|\nabla u(t)\|_2\|u(t)\|_2^{\frac{1-s_c}{s_c}} > x_1$ for all time $t \in I$, which proves (2.11).

□

The following is another statement of the dichotomy proposition in terms of λ and $\eta(t)$ defined by (2.6) and (2.4) respectively, which will be useful in the sequel.

Proposition 2.2. *Let* $M(u)^{\frac{1-s_c}{s_c}}E(u) < M(Q)^{\frac{1-s_c}{s_c}}E(Q)$ *and* $0 \leq \lambda_- < 1 < \lambda$ *be defined by* (2.6). *Then exactly one of the following holds:*
(1) The solution $u(t)$ *to*(1.1) *is global and*

$$\frac{1}{2\omega_1}\frac{E(u)M(u)^{\frac{1-s_c}{s_c}}}{E(Q)M(Q)^{\frac{1-s_c}{s_c}}} \leq \eta(t)^2 \leq \lambda_-^2, \quad \forall\ t \in (-\infty, +\infty)$$

(2) $1 < \lambda^2 \leq \eta(t)^2$, $\forall\ t \in (T_-, T_+)$.

Naturally, whether the solution is of the first or second type in Proposition 2.2 is determined by checking the initial data. Note that the second case does not assert finite-time blow-up. In the first case, we have further results as follows , the proof of which is almost the same as [18].

Lemma 2.3. *(Small initial data). Let $\|u_0\|_{\dot H^{s_c}} \le A$, then there exists $\delta_{sd} = \delta_{sd}(A) > 0$ such that if $\|e^{it\Delta}u_0\|_{S(\dot H^{s_c})} \le \delta_{sd}$, then u solving (1.1) is global and*

$$\|u\|_{S(\dot H^{s_c})} \le 2\|e^{it\Delta}u_0\|_{S(\dot H^{s_c})}, \tag{2.13}$$

$$\|D^{s_c}u\|_{S(L^2)} \le 2c\|u_0\|_{\dot H^{s_c}}. \tag{2.14}$$

(one will find $\|\cdot\|_{S(\dot H^{s_c})}$ in Section 6, and note that by Strichartz estimates, the hypotheses are satisfied if $\|u_0\|_{\dot H^{s_c}} \le C\delta_{sd}$.)

Lemma 2.4. *(Existence of wave operators). Suppose that $\psi^+ \in H^1$ and*

$$\frac{1}{2}||\nabla\psi^+||_2^2 M(\psi^+)^{\frac{1-s_c}{s_c}} < E(Q)M(Q)^{\frac{1-s_c}{s_c}}. \tag{2.15}$$

Then there exists $v_0 \in H^1$ such that v solves (1.1) with initial data v_0 globally in H^1 with

$$\|\nabla v(t)\|_2\|v_0\|_2^{\frac{1-s_c}{s_c}} < \|\nabla Q\|_2\|Q\|_2^{\frac{1-s_c}{s_c}}, M(v) = \|\psi^+\|_2^2, E[v] = \frac{1}{2}\|\nabla\psi^+\|_2^2,$$

and

$$\lim_{t\to+\infty}\|v(t) - e^{it\Delta}\psi^+\|_{H^1} = 0.$$

Moreover, if $\|e^{it\Delta}\psi^+\|_{S(\dot H^{s_c})} \le \delta_{sd}$, then

$$\|v_0\|_{\dot H^{s_c}} \le 2\|\psi^+\|_{\dot H^{s_c}} \text{ and } \|v\|_{S(\dot H^{s_c})} \le 2\|e^{it\Delta}\psi^+\|_{S(\dot H^{s_c})}.$$

$$\|D^s v\|_{S(L^2)} \le c\|\psi^+\|_{\dot H^s}, 0 \le s \le 1.$$

Theorem 2.5. *(Scattering). If $0 < M(u)^{\frac{1-s_c}{s_c}}E(u)/M(Q)^{\frac{1-s_c}{s_c}}E(Q) < 1$ and the first case of Proposition 2.2 holds, then $u(t)$ scatters as $t \to +\infty$ or $t \to -\infty$. That means there exist $\phi_\pm \in H^1$ such that*

$$\lim_{t\to\pm\infty}\|u(t) - e^{-it\Delta}\phi_\pm\|_{H^1} = 0. \tag{2.16}$$

Consequently, we have that

$$\lim_{t\to\pm\infty}\|u(t)\|_{L^{p+1}} = 0 \tag{2.17}$$

and

$$\lim_{t\to\pm\infty}\eta(t)^2 = \frac{1}{2\omega_1}\frac{E(u)M(u)^{\frac{1-s_c}{s_c}}}{E(Q)M(Q)^{\frac{1-s_c}{s_c}}}. \tag{2.18}$$

3. Virial Identity and Blow-Up Conditions

In the sequel we focus on the second case of Proposition 2.2. Using the classical virial identity, we first derive the upper bound on the finite blow-up time under the finite variance hypothesis.

Proposition 3.1. *Let $M(u) = M(Q)$, $E(u)^{s_c} < E(Q)^{s_c}$. Suppose $\|xu_0\|_2 < +\infty$ and suppose the second case of Proposition 2.2 holds ($\lambda > 1$ is defined by (2.6)).Define $r(t)$ to be the scaled variance:*

$$r(t) = \frac{\|xu\|_2^2}{\left(-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}\right)E(Q)}$$

Then blow-up occurs in forward time before t_b, where

$$t_b = r'(0) + \sqrt{r'(0)^2 + 2r(0)}.$$

Note that

$$r(0) = \frac{\|xu_0\|_2^2}{\left(-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}\right)E(Q)}$$

and

$$r'(0) = \frac{Im\int(x\cdot\nabla u_0)\overline{u_0}}{\left(-4\omega_1\lambda^2 + N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}\right)E(Q)}.$$

Proof. The virial identity gives

$$r''(t) = \frac{4N(p-1)E(u) - (2N(p-1)-8)\,\|\nabla u\|_2^2}{\left(-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}\right)E(Q)}.$$

Identities (2.2) imply

$$r''(t) = \frac{4N(p-1)\frac{E(u)}{E(Q)} - 2\omega_1\left(2N(p-1)-8\right)\frac{\|\nabla u\|_2^2}{\|\nabla Q\|_2^2}}{-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}}.$$

By the definition of λ and η,

$$r''(t) = \frac{4N(p-1)(\omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}) - 2\omega_1\left(2N(p-1)-8\right)\eta(t)^2}{-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}}.$$

Since $\eta(t) \geq \lambda > 1$, we have

$$r''(t) \leq -1,$$

which, by integrating in time twice, gives

$$r(t) \leq -\frac{1}{2}t^2 + r'(0)t + r(0).$$

The positive root of the polynomial on the right hand side is t_b given in the proposition statement.

□

The next result is related to the local virial identity. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be radial such that

$$\varphi(x) = \begin{cases} |x|^2, & |x| \leq 1; \\ 0, & |x| \geq 2. \end{cases}$$

For $R > 0$ define

$$z_R(t) = \int R^2 \phi(\frac{x}{R})|u(x,t)|^2 dx. \tag{3.1}$$

Then we can directly calculate the following local virial identity:

$$\begin{aligned} z_R''(t) &= 4\int \partial_j\partial_k\phi(\frac{x}{R})\partial_j u \partial_k \bar{u} dx - \int \Delta\phi(\frac{x}{R})|u|^4 dx - \frac{1}{R^2}\int \Delta^2\phi(\frac{x}{R})|u|^2 dx \\ &= \left(4N(p-1)E(u) - (2N(p-1)-8)\,\|\nabla u\|_2^2\right) + A_R(u(t)), \end{aligned} \tag{3.2}$$

where for a constant C_1 we can control

$$A_R(u(t)) \leq C_1\left(\frac{1}{R^2}\|u\|^2_{L^2(|x|\geq R)} + \|u\|^{p+1}_{L^{p+1}(|x|\geq R)}\right). \tag{3.3}$$

The local virial identity will give another version of Proposition 3.1 , for which , without the assumption of finite variance, we will assumes that the solution is suitably localized in H^1 for all times. Define

$$\eta_{\geq R} = \frac{\|u\|^{s_c(p-1)}_{L^2(|x|\geq R)}\|\nabla u\|^{(1-s_c)(p-1)}_{L^2(|x|\geq R)}}{\|Q\|_2^{s_c(p-1)}\|\nabla Q\|_2^{(1-s_c)(p-1)}}.$$

Proposition 3.2. *Let $M(u) = M(Q)$, $E(u) < E(Q)$ and suppose the second case of Proposition 2.2 holds ($\lambda > 1$ is defined in (2.6)). Select γ such that*

$$0 < \gamma < min\left(2\omega_1\left(2N(p-1)-8\right), 4N(p-1)\omega_2\lambda^{\frac{N(p-1)-4}{2}} - 16\omega_1\right).$$

Suppose that there is a radius $R \geq C_2\gamma^{-\frac{1}{2}}$ such that for all t, there holds $\eta_{\geq R} \leq \gamma$. Define $\tilde{r}(t)$ to be the scaled local variance:

$$\tilde{r}(t) = \frac{z_R(t)}{CE(Q)\left(-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}} - \gamma\lambda^2\right)}$$

(C is an absolute constant determined by C_1 and C_2). Then blow-up occurs in forward time before t_b, where

$$t_b = \tilde{r}'(0) + \sqrt{\tilde{r}'(0)^2 + 2\tilde{r}(0)}.$$

Proof. By the local virial identity and the same steps in the proof of Proposition 3.1

$$\tilde{r}''(t) = \frac{1}{C}\frac{4N(p-1)(\omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}) - 2\omega_1\left(2N(p-1)-8\right)\eta(t)^2 + A_R(u(t))/E(Q)}{-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}} - \gamma\lambda^2}$$

By the exterior Gagliardo-Nirenberg inequality, we have

$$\|u\|^{p+1}_{L^{p+1}(|x|\geq R)} \leq C_{GN}\|\nabla u\|^{\frac{N(p-1)}{2}}_{L^2(|x|\geq R)}\|u\|^{2-\frac{(N-2)(p-1)}{2}}_{L^2(|x|\geq R)} \leq \|\nabla u\|_2^2\eta_{\geq R} \leq \|\nabla Q\|_2^2\gamma\eta(t)^2. \quad (3.4)$$

This combined with

$$\frac{1}{R^2}\|u\|^2_{L^2(|x|\geq R)} \leq C_2^{-2}\|Q\|_2^2\gamma \leq C_2^{-2}\|Q\|_2^2\gamma\eta(t)^2 \quad (3.5)$$

gives

$$\begin{aligned}\tilde{r}''(t) &\leq \frac{1}{C}\frac{4N(p-1)(\omega_1\lambda^2-\omega_2\lambda^{\frac{N(p-1)}{2}}) - 2\omega_1\left(2N(p-1)-8\right)\eta(t)^2 + C_3\frac{(\|Q\|_2^2+\|\nabla Q\|_2^2)}{E(Q)}\gamma\eta(t)^2}{-16\omega_1\lambda^2+4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}-\gamma\lambda^2} \\ &\leq \frac{1}{C}\frac{C_4\left(4N(p-1)(\omega_1\lambda^2-\omega_2\lambda^{\frac{N(p-1)}{2}}) - 2\omega_1\left(2N(p-1)-8\right)\eta(t)^2 + \gamma\eta(t)^2\right)}{-16\omega_1\lambda^2+4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}}-\gamma\lambda^2}.\end{aligned}$$

Taking the constant $C = C_4$, since $\eta(t) \geq \lambda > 1$ and from the selection of γ, we obtain

$$r''(t) \leq -1.$$

The remainder of the argument is the same as in the proof of Proposition 3.1 .

□

Finally, we will give the finite blow-up time for radial solutions before which we would like to introduce the Radial Gagliardo-Nirenberb inequality:

Lemma 3.3. [16] *(Radial Gagliardo-Nirenberb inequality). For all $\delta > 0$, there exists a constant $C_\delta > 0$ such that for all $u \in \dot{H}^{s_c}$ with radial symmetry, and for all $R > 0$, we have*

$$\int_{|x|\geq R}|u|^{p+1}dx \leq \delta\int_{|x|\geq R}|\nabla u|^2dx + \frac{C_\delta}{R^{2(1-s_c)}}\left[(\rho(u,R))^{\frac{2(p+3)}{5-p}} + (\rho(u,R))^{\frac{p+1}{2}}\right],$$

where $\rho(u,R) = \sup_{R'\geq R}\frac{1}{(R')^{2s_c}}\int_{R'\leq|x|\leq 2R'}|u|^2dx$.

Note that this lemma implies that for all $\delta > 0$, there exists a constant $C_\delta > 0$ and $C_Q > 0$ such that for all $u \in \dot{H}^{s_c}$ with radial symmetry and $M(u) = M(Q)$, and for all $R > 0$, we have

$$\int_{|x|\geq R}|u|^{p+1}dx \leq \delta\int_{|x|\geq R}|\nabla u|^2dx + \frac{C_\delta C_Q}{R^{2(1-s_c)}}. \quad (3.6)$$

Proposition 3.4. *Let $M(u) = M(Q)$, $E(u) < E(Q)$ and suppose the second case of Proposition 2.2 holds ($\lambda > 1$ is defined in* (2.6)*.) Suppose that u is radial. Select γ such that*

$$0 < \gamma < min\left(2\omega_1\left(2N(p-1)-8\right), 4N(p-1)\omega_2\lambda^{\frac{N(p-1)-4}{2}} - 16\omega_1\right).$$

Then for

$$R > \max\left(\gamma^{-\frac{1}{2}}, \left(\frac{2C_\gamma}{-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}} - \gamma\lambda^2}\right)^{\frac{1}{2(1-s_c)}}\right)$$

we define $\tilde{r}(t)$ to be the scaled local variance:

$$\tilde{r}(t) = \frac{z_R(t)}{\tilde{C}_Q E(Q)\left(-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}} - \gamma\lambda^2\right)}$$

(where the constant $\tilde{C}_Q$ is dependent on Q determined by C_γ and C_Q in (3.6)*). Then blow-up occurs in forward time before t_b, where*

$$t_b = \tilde{r}'(0) + \sqrt{\tilde{r}'(0)^2 + 2\tilde{r}(0)}.$$

Proof. We modify the proof of Proposition 3.2 only in (3.4) and (3.5). From the Radial Gagliardo-Nirenberb inequality (3.6) with $\delta = \gamma$, we obtain

$$\|u\|_{L^{p+1}(|x|\geq R)}^{p+1} \leq C_Q\left(\gamma\eta(t)^2 + \frac{C_\gamma}{R^{2(1-s_c)}}\right).$$

If taking C_Q to stand for the variant constants dependent on Q, we have

$$\frac{1}{R^2}\|u\|_{L^2(|x|\geq R)}^2 \leq \frac{C_Q}{R^2} \leq \frac{C_Q\eta(t)^2}{R^2} \leq C_Q\gamma\eta(t)^2.$$

Thus

$$\begin{aligned}\tilde{r}''(t) &\leq C_Q\frac{4N(p-1)(\omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}) - 2\omega_1\left(2N(p-1)-8\right)\eta(t)^2 + \gamma\eta(t)^2 + \frac{C_\gamma}{R^{2(1-s_c)}}}{\tilde{C}_Q\left(-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}} - \gamma\lambda^2\right)} \\ &\leq C_Q\frac{\left(4N(p-1)(\omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}) - 2\omega_1\left(2N(p-1)-8\right)\eta(t)^2 + \gamma\eta(t)^2\right) + \frac{C_\gamma}{R^{2(1-s_c)}}}{\tilde{C}_Q\left(-16\omega_1\lambda^2 + 4N(p-1)\omega_2\lambda^{\frac{N(p-1)}{2}} - \gamma\lambda^2\right)}.\end{aligned}$$

We only have to select $\tilde{C}_Q = 2C_Q$ in the assumptions. Then since $\eta(t) \geq \lambda > 1$, the restriction of γ and R gives

$$r''(t) \leq -1,$$

and we conclude the proof with the same steps as in the proof of Proposition 3.1. □

4. Variational Characterization of the Ground State

This section deals with the variation characterization of Q stated in the above section. It is an important preparation for the "near boundary case" in Section 5. For now, we will write $u = u(x)$ as the time dependence plays no role in what follows.

Proposition 4.1. *There exists a function $\epsilon(\rho)$ with $\epsilon(\rho) \to 0$ as $\rho \to 0$ such that the following holds: suppose there is $\lambda > 0$ satisfying*

$$\left| \frac{M(u)^{\frac{1-s_c}{s_c}} E(u)}{M(Q)^{\frac{1-s_c}{s_c}} E(Q)} - \left(\omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}} \right) \right| \le \rho \lambda^{\frac{N(p-1)}{2}}, \tag{4.1}$$

and

$$\left| \frac{\|u\|_2^{\frac{1-s_c}{s_c}} \|\nabla u\|_2}{\|Q\|_2^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2} - \lambda \right| \le \rho \begin{cases} \lambda, & \lambda \ge 1 \\ \lambda^2, & \lambda \le 1. \end{cases} \tag{4.2}$$

Then there exists $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$ such that

$$\left\| u - e^{i\theta} \lambda^{\frac{N}{2}} \beta^{-\frac{2}{p-1}} Q\left(\lambda(\beta^{-1} \cdot - x_0)\right) \right\|_2 \le \beta^{\frac{N}{2} - \frac{2}{p-1}} \epsilon(\rho) \tag{4.3}$$

and

$$\left\| \nabla \left[u - e^{i\theta} \lambda^{\frac{N}{2}} \beta^{-\frac{2}{p-1}} Q\left(\lambda(\beta^{-1} \cdot - x_0)\right) \right] \right\|_2 \le \lambda \beta^{\frac{N}{2} - \frac{2}{p-1} - 1} \epsilon(\rho), \tag{4.4}$$

where $\beta = \left(\frac{M(u)}{M(Q)}\right)^{\frac{p-1}{N(p-1)-4}}$.

Remark 4.2. If we let $v(x) = \beta^{\frac{2}{p-1}} u(\beta x)$, then $M(v) = \beta^{\frac{4}{p-1} - N} M(u) = M(Q)$, and we can then restate Proposition 4.1 as follows:

Suppose $\|v\|_2 = \|Q\|_2$ and there is $\lambda > 0$ such that

$$\left| \frac{E(v)}{E(Q)} - \left(\omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}} \right) \right| \le \rho \lambda^{\frac{N(p-1)}{2}}, \tag{4.5}$$

and

$$\left| \frac{\|\nabla v\|_2}{\|\nabla Q\|_2} - \lambda \right| \le \rho \begin{cases} \lambda, & \lambda \ge 1 \\ \lambda^2, & \lambda \le 1. \end{cases} \tag{4.6}$$

Then there exists $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$ such that

$$\left\| v - e^{i\theta} \lambda^{\frac{N}{2}} Q\left(\lambda(\cdot - x_0)\right) \right\|_2 \le \epsilon(\rho) \tag{4.7}$$

and

$$\left\| \nabla \left[v - e^{i\theta} \lambda^{\frac{N}{2}} Q\left(\lambda(\cdot - x_0)\right) \right] \right\|_2 \le \lambda \epsilon(\rho). \tag{4.8}$$

Thus it suffices to prove the scaled statement equivalent to Proposition 4.1 and we will carry it out by means of the following result from Lions [14].

Proposition 4.3. *([14]) There exists a function $\epsilon(\rho)$, defined for small $\rho > 0$ such that* $\lim_{\rho \to 0} \epsilon(\rho) = 0$, *such that for all $u \in H^1$ with*

$$\big| \|u\|_{p+1} - \|Q\|_{p+1} \big| + \big| \|u\|_2 - \|Q\|_2 \big| + \big| \|\nabla u\|_2 - \|\nabla Q\|_2 \big| \le \rho, \tag{4.9}$$

there exist $\theta_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$ such that

$$\left\| u - e^{i\theta_0} Q(\cdot - x_0) \right\|_{H^1} \le \epsilon(\rho). \tag{4.10}$$

Proof. (Proof of proposition 4.1). As a result of Remark 4.2, we will just prove the equivalent version rescaling off the mass. Set $\tilde{u}(x) = \lambda^{-\frac{N}{2}} v(\lambda^{-1}x)$, and then (4.6) gives

$$\left| \frac{\|\nabla \tilde{u}\|_2}{\|\nabla Q\|_2} - 1 \right| \leq \rho. \tag{4.11}$$

On the other hand, by (2.2) and the notation of ω_1 and ω_2 we have

$$\begin{aligned} \left| \frac{\|v\|_{p+1}^{p+1}}{\|Q\|_{p+1}^{p+1}} - \lambda^{\frac{N(p-1)}{2}} \right| &\leq \left| -\frac{1}{\omega_2} \left(\frac{E(v)}{E(Q)} - (\omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}}) \right) \right| + \left| \frac{N(p-1)}{4} \frac{\|\nabla v\|_2^2}{\|\nabla Q\|_2^2} - \frac{\omega_1}{\omega_2} \lambda^2 \right| \\ &= \frac{1}{\omega_2} \left| \frac{E(v)}{E(Q)} - (\omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}}) \right| + \frac{N(p-1)}{4} \left| \frac{\|\nabla v\|_2^2}{\|\nabla Q\|_2^2} - \lambda^2 \right|. \end{aligned}$$

Then (4.5) and (4.6) imply

$$\begin{aligned} \left| \frac{\|v\|_{p+1}^{p+1}}{\|Q\|_{p+1}^{p+1}} - \lambda^{\frac{N(p-1)}{2}} \right| &\leq \frac{1}{\omega_2} \rho \lambda^{\frac{N(p-1)}{2}} + \frac{N(p-1)}{4} \rho \begin{cases} \lambda^2, & \lambda \geq 1 \\ \lambda^4, & \lambda \leq 1 \end{cases} \\ &\leq (\frac{N(p-1)}{2} - 1) \rho \lambda^{\frac{N(p-1)}{2}}. \end{aligned}$$

Thus in terms of $\tilde{u}$, we obtain

$$\left| \frac{\|\tilde{u}\|_{p+1}^{p+1}}{\|Q\|_{p+1}^{p+1}} - 1 \right| \leq \frac{N(p-1) - 2}{2} \rho. \tag{4.12}$$

Thus (4.11) and (4.12) imply that the condition (4.9) is satisfied by $\tilde{u}$. By Proposition 4.3 and rescaling back to v, we obtain (4.7) and (4.8) .

□

5. Near-Boundary Case

We know from Proposition 2.2 that if $M(u) = M(Q)$ and $E(u)/E(Q) = \omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}}$ for some $\lambda > 1$ and $\|\nabla u_0\|_2 / \|\nabla Q\|_2 \geq \lambda$, then $\|\nabla u(t)\|_2 / \|\nabla Q\|_2 \geq \lambda$ for all t. Now in this section, we will claim that $\|\nabla u(t)\|_2 / \|\nabla Q\|_2$ cannot remain near λ globally in time.

Proposition 5.1. *Let $\lambda_0 > 1$. There exists $\rho_0 = \rho_0(\lambda_0) > 0$ with the property that $\rho_0(\lambda_0) \to 0$ as $\lambda_0 \to 1$, such that for any $\lambda \geq \lambda_0$, the following holds: There does not exist a solution $u(t)$ of problem* (1.1) *with $P(u) = 0$ satisfying $M(u) = M(Q)$,*

$$\frac{E(u)}{E(Q)} = \omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}}, \tag{5.1}$$

and for all $t \geq 0$

$$\lambda \leq \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2} \leq \lambda(1 + \rho_0). \tag{5.2}$$

We would like to give another equivalent statement implied by this assertion: For any solution $u(t)$ to (1.1) with $P(u) = 0$ satisfying $M(u) = M(Q)$,

$$\frac{E(u)}{E(Q)} = \omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}},$$

and for all $t \geq 0$

$$\lambda \leq \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2},$$

there exist a time $t_0 \geq 0$ such that

$$\frac{\|\nabla u(t_0)\|_2}{\|\nabla Q\|_2} \geq \lambda(1 + \rho_0).$$

Before proving Proposition 5.1 we will firstly give a useful lemma the proof of which will be found in [8].

Lemma 5.2. *Suppose that $u(t)$ with $P(u) = 0$ solving (1.1) satisfies, for all t*

$$\left\| u(t) - e^{i\theta(t)} Q(\cdot - x(t)) \right\|_{H^1} \leq \epsilon \tag{5.3}$$

for some continuous functions $\theta(t)$ and $x(t)$. Then

$$\frac{|x(t)|}{t} \leq C\epsilon^2 \quad as \quad t \to +\infty.$$

Proof. (Proof of proposition 5.1). To the contrary, we suppose that there exists a solution $u(t)$ satisfying $M(u) = M(Q)$, $E(u)/E(Q) = \omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}}$ and

$$\lambda \leq \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2} \leq \lambda(1 + \rho_0). \tag{5.4}$$

Since $\|\nabla u(t)\|_2^2 \geq \lambda^2 \|\nabla Q\|_2^2 = 2\omega_1 \lambda^2 E(Q)$, we have

$$\begin{aligned} &4N(p-1)E(u) - (2N(p-1) - 8) \|\nabla u\|_2^2 \\ \leq &- 4 \left(N(p-1)\omega_2 \lambda^{\frac{N(p-1)}{2}} - 4\omega_1 \lambda^2 \right) E(Q). \end{aligned}$$

By Proposition 4.1, there exist functions $\theta(t)$ and $x(t)$ such that for $\rho = \rho_0$

$$\left\| u(t) - e^{i\theta(t)} \lambda^{\frac{N}{2}} Q\left(\lambda(\cdot - x(t))\right) \right\|_2 \leq \epsilon(\rho) \tag{5.5}$$

and

$$\left\| \nabla \left[u(t) - e^{i\theta(t)} \lambda^{\frac{N}{2}} Q\left(\lambda(\cdot - x(t))\right) \right] \right\|_2 \leq \lambda \epsilon(\rho). \tag{5.6}$$

By the continuity of the $u(t)$ flow, we may assume $\theta(t)$ and $x(t)$ are continuous. Let

$$R(T) = \max \left(\max_{0 \leq t \leq T} |x(t)|, \log \epsilon(\rho)^{-1} \right).$$

For fixed T, take $R = R(T)$ in the local virial identity (3.2) . Then owing to the exponential localization of $Q(x)$, (5.5) and (5.6) imply that,

$$|A_R(u(t))| \le \frac{C}{2}\lambda^2 \left(\epsilon(\rho) + e^{-R(T)}\right)^2 \le C\lambda^2\epsilon(\rho)^2.$$

Taking $\rho = \rho_0$ small enough to make $\epsilon(\rho)$ small such that for all $0 \le t \le T$,

$$z_R''(t) \le -CE(Q)(\lambda^{\frac{N(p-1)}{2}} - \lambda^2),$$

and so

$$\frac{z_R(T)}{T^2} \le \frac{z_R(0)}{T^2} + \frac{z_R'(0)}{T} - CE(Q)(\lambda^{\frac{N(p-1)}{2}} - \lambda^2).$$

By definition of $z_R(t)$ we have

$$|z_R(0)| \le CR^2\|u_0\|_2^2 = C\|Q\|_2^2R^2$$

and

$$|z_R'(0)| \le CR\|u_0\|_2\|\nabla u_0\|_2 \le C\|Q\|_2\|\nabla Q\|_2R(1+\rho_0)\lambda.$$

Consequently,

$$\frac{z_{2R(T)}(T)}{T^2} \le C\left(\frac{R(T)^2}{T^2} + \frac{\lambda R(T)}{T}\right) - CE(Q)(\lambda^{\frac{N(p-1)}{2}} - \lambda^2).$$

Taking T sufficiently large ,Lemma 5.2 implies

$$0 \le \frac{z_{2R(T)}(T)}{T^2} \le C\left(\lambda\epsilon(\rho)^2 - (\lambda^{\frac{N(p-1)}{2}} - \lambda^2)\right) < 0$$

provided taking ρ_0 small enough .

Note that ρ_0 is independent of T . We then get a contradiction.

□

6. Profile Decomposition

In this section we make some extension of the cubic profile decomposition [8] to our general case, and we review some work done by the author in [18].

First of all , we introduce some notations. We say that (q, r) is $\dot{H}^s(\mathbb{R}^N)$ admissible and denote it by $(q, r) \in \Lambda_s$ if

$$\frac{2}{q} + \frac{N}{r} = \frac{N}{2} - s, \quad \frac{2N}{N-2s} < r < \frac{2N}{N-2}$$

Correspondingly, we denote (q', r') the dual $\dot{H}^s(\mathbb{R}^N)$ admissible by $(q', r') \in \Lambda_s'$ if $(q, r) \in \Lambda_{-s}$ with (q', r') is the Hölder dual to (q, r). We also define the following Srichartz norm

$$\|u\|_{S(\dot{H}^s)} = \sup_{(q,r)\in\Lambda_s} \|u\|_{L_t^qL_x^r}$$

and the dual Strichartz norm

$$\|u\|_{S'(\dot{H}^{-s})} = \inf_{(q',r')\in\Lambda_s'} \|u\|_{L_t^{q'}L_x^{r'}} = \inf_{(q,r)\in\Lambda_{-s}} \|u\|_{L_t^{q'}L_x^{r'}},$$

where (q', r') is the Hölder dual to (q, r). Also as in [8] , the notation $S(\dot{H}^s; I)$ and $S'(\dot{H}^s; I)$ indicate a restriction to a time subinterval $I \subset (-\infty, +\infty)$.

Remark 6.1. By notation $\|\cdot\|_{S(\dot{H}^{s_c})}$ in the sequel, we will in fact add the restriction $q \geq r$ to the definition of $(q, r) \in \Lambda_{s_c}$ without affecting the future arguments for our main results in this paper, which is needed in the proof of Lemma 6.7 below.

Now we first restate the linear profile decomposition below which was shown in [18] .

Lemma 6.2. *(Profile expansion). Let $\phi_n(x)$ be an uniformly bounded sequence in H^1, then for each M there exists a subsequence of ϕ_n, also denoted by ϕ_n, and (1) for each $1 \leq j \leq M$, there exists a (fixed in n) profile $\tilde{\psi}^j(x)$ in H^1, (2) for each $1 \leq j \leq M$, there exists a sequence(in n)of time shifts t_n^j, (3) for each $1 \leq j \leq M$, there exists a sequence (in n) of space shifts x_n^j, (4) there exists a sequence (in n) of remainders $\tilde{W}_n^M(x)$ in H^1, such that*

$$\phi_n(x) = \sum_{j=1}^{M} e^{-it_n^j\Delta}\tilde{\psi}^j(x - x_n^j) + \tilde{W}_n^M(x),$$

The time and space sequences have a pairwise divergence property, i.e., for $1 \leq j \neq k \leq M$, we have

$$\lim_{n\to+\infty}(|t_n^j - t_n^k| + |x_n^j - x_n^k|) = +\infty.$$

The remainder sequence has the following asymptotic smallness property:

$$\lim_{M\to+\infty}[\lim_{n\to+\infty}\|e^{it\Delta}\tilde{W}_n^M\|_{S(\dot{H}^{s_c})}] = 0.$$

For fixed M and any $0 \leq s \leq 1$, we have the asymptotic Pythagorean expansion:

$$\|\phi_n\|_{\dot{H}^s}^2 = \sum_{j=1}^{M}\|\tilde{\psi}^j\|_{\dot{H}^s}^2 + \|\tilde{W}_n^M\|_{\dot{H}^s}^2 + o_n(1).$$

Remark 6.3. We omit the proof of Lemma 6.2, but would like to point out some modification from the statement in [18]: In the reference the author introduced a concept of k-point $(\frac{1}{r}, \frac{1}{q})$ for which one can also refer to [9], and gave the proof in terms of that conception. In fact, it is easy to check that $(\frac{1}{r}, \frac{1}{q})$ is a p-point with the same p in our equation (1.1) , if and only if $(q, r) \in \Lambda_{s_c}$. Moreover, it is interesting to note that if $(q, r) \in \Lambda_{s_c}$ then $(\frac{q}{p}, \frac{r}{p}) \in \Lambda'_{s_c}$. Thus we have the following Strichartz estimate which was frequently used in [18]:

$$\left\|i\int_0^t e^{i(t-t')\Delta}|u|^{p-1}|u|(x,t')dt'\right\|_{L_t^qL_x^r} \leq C\left\||u|^{p-1}|u|\right\|_{L_t^{\frac{q}{p}}L_x^{\frac{r}{p}}} \leq C\left\|u\right\|_{L_t^qL_x^r}^p.$$

Furthermore, the author in [18] gave another useful claim and we will restate the equivalent version as follows: For any $(q, r) \in \Lambda_{s_c}$, there exists $(q_1, r_1) \in \Lambda_0$ and $(q_2', r_2') \in \Lambda_0'$ such that

$$\begin{cases} \frac{1}{q_2'} = \frac{1}{q_1} + \frac{p-1}{q} \\ \frac{1}{r_2'} = \frac{1}{r_1} + \frac{p-1}{r}. \end{cases}$$

Applying the above observation, our proof of Lemma 6.2 will be almost the same as that in [18], and that is why we will omit it here.

Similar to Keraani [12] and [10], we give the following definition of the nonlinear profile:

Definition 6.4. Let V be a solution to the linear Schrödinger equation. We say U is the nonlinear profile associated to $(V, \{t_n\})$, if U is a solution to the Hartree equation (1.1) satisfying

$$\|(U-V)(-t_n)\|_{H^1} \to 0 \quad as \quad n \to \infty.$$

Note that, similar to the arguments in [10], by the local theory and the proof of the existence of wave operators, there always exist a nonlinear profile associated to a given $(V, \{t_n\})$. Thus for every j, there exists a solution v^j to (1.1) associated to $(\tilde{\psi}^j, \{t_n^j\})$ such that

$$\|v^j(\cdot - x_n^j, -t_n^j) - e^{-it_n^j\Delta}\tilde{\psi}^j(\cdot - x_n^j)\|_{H^1} \to 0 \quad as \quad n \to \infty.$$

If we let $NLH(t)\psi$ denote the solution to (1.1) with initial data ψ, by shifting the linear profile $\tilde{\psi}^j$ when necessary, we may denote $v^j(-t_n^j)$ as $NLH(-t_n^j)\psi^j$ with some $\psi^j \in H^1$. Thus using the same method of replacing linear flows by nonlinear flows as applied in [7] to give the following proposition:

Proposition 6.5. *Let $\phi_n(x)$ be an uniformly bounded sequence in H^1 . There exists a subsequence of ϕ_n , also denoted by ϕ_n , profiles $\psi^j(x)$ in H^1 , and parameters x_n^j , t_n^j so that for each M ,*

$$\phi_n(x) = \sum_{j=1}^{M} NLS(-t_n^j)\psi^j(x - x_n^j) + W_n^M(x), \tag{6.1}$$

where as $n \to \infty$
- *For each j , either $t_n^j = 0$, $t_n^j \to +\infty$ or $t_n^j \to -\infty$.*
- *If $t_n^j \to +\infty$, then $\|NLS(-t)\psi^j\|_{S(\dot{H}^{s_c};[0,\infty))} < \infty$ and if $t_n^j \to -\infty$, then $\|NLS(-t)\psi^j\|_{S(\dot{H}^{s_c};[-\infty,0))} < \infty$*
- *For $j \neq k$,*
$$\lim_{n\to+\infty}(|t_n^j - t_n^k| + |x_n^j - x_n^k|) = +\infty.$$
- *$NLS(t)W_n^M$ is global for M large enough with*
$$\lim_{M\to+\infty}[\lim_{n\to+\infty}\|NLS(t)W_n^M\|_{S(\dot{H}^{s_c})}] = 0.$$

We also have the H^s Pythagorean decomposition: for fixed M and $0 \le s \le 1$,

$$\|\phi_n\|_{\dot{H}^s}^2 = \sum_{j=1}^{M} \|NLS(-t_n^j)\psi^j\|_{\dot{H}^s}^2 + \|W_n^M\|_{\dot{H}^s}^2 + o_n(1), \tag{6.2}$$

and the energy Pythagorean decomposition

$$E(\phi_n) = \sum_{j=1}^{M} E(\psi^j) + E(W_n^M) + o_n(1). \tag{6.3}$$

From a similar argument in [7], we know that (6.3) was proven by establishing the following first

$$\|\phi_n\|_{p+1}^{p+1} = \sum_{j=1}^{M} \|NLS(-t_n^j)\psi^j\|_{p+1}^{p+1} + \|W_n^M\|_{p+1}^{p+1} + o_n(1). \tag{6.4}$$

The next lemma is an extension of the perturbation theory for the case $N = 3$ [7]. By virtue of Remark 6.3, the proof will also be similar to [18], which we will represent in this paper.

Lemma 6.6. *(Perturbation Theory). For each $A \geq 1$, there exists $\epsilon_0 = \epsilon_0(A) \ll 1$ and $c = c(A)$ such that the following holds: Fix $T > 0$. Let $u = u(x,t) \in L^\infty([0,T]; H^1)$ solve*

$$iu_t + \Delta u + |u|^{p-1}u = 0$$

on $[0,T]$. Let $\tilde{u} = \tilde{u}(x,t) \in L^\infty([0,T]; H^1)$ and define

$$e = i\tilde{u}_t + \Delta\tilde{u} + |\tilde{u}|^{p-1}\tilde{u}.$$

For each $\epsilon \leq \epsilon_0$, if for some $(q_1, r_1) \in \Lambda_{-s_c}$

$$\|\tilde{u}\|_{S(\dot{H}^{s_c};[0,T])} \leq A, \quad \|e\|_{S'(\dot{H}^{-s_c};[0,T])} \leq \epsilon, \quad \text{and} \quad \|e^{it\Delta}(u(0) - \tilde{u}(0))\|_{S(\dot{H}^{s_c};[0,T])} \leq \epsilon,$$

then

$$\|u - \tilde{u}\|_{S(\dot{H}^{s_c};[0,T])} \leq c(A)\epsilon.$$

Proof. Under the condition of the lemma, it suffices to prove that for any $(q,r) \in \Lambda_{s_c}$ and for some $(q_1, r_1) \in \Lambda_{-s_c}$, if

$$\|\tilde{u}\|_{L_t^q L_x^r} \leq A, \quad \|e\|_{L_t^{q_1'} L_x^{r_1'}} \leq \epsilon, \quad \text{and} \quad \|e^{it\Delta}(u(0) - \tilde{u}(0))\|_{L_t^q L_x^r} \leq \epsilon,$$

then

$$\|u - \tilde{u}\|_{L_t^q L_x^r} \leq c(A)\epsilon.$$

In fact, the following arguments are similar to that in [18] except for some slight differences. One can also refer to [7] for a similar proof.

Let w defined by $u = \tilde{u} + w$, then w solves

$$iw_t + \Delta w + |\tilde{u} + w|^{p-1}(\tilde{u} + w) - |\tilde{u}|^{p-1}\tilde{u} + e = 0. \tag{6.5}$$

Since $\|\tilde{u}\|_{L_t^q L_x^r} \leq A$, we can partition $[0,T]$ into $N = N(A)$ intervals $I_j = [t_j, t_{j+1}]$ such that for every j, $\|\tilde{u}\|_{L_{t\in I_j}^q L_x^r} \leq \delta$ with δ sufficiently small to be specified later. The integral equation of (6.5) with initial data $w(t_j)$ is

$$w(t) = e^{i(t-t_j)\Delta}w(t_j) + i\int_{t_j}^{t} e^{i(t-s)\Delta}W(\cdot, s)ds, \tag{6.6}$$

where

$$W = (-|\tilde{u} + w|^{p-1}(\tilde{u} + w) + |\tilde{u}|^{p-1}\tilde{u}) - e.$$

Applying the inhomogeneous Strichartz estimate in I_j and from Remark 6.3, we have

$$\begin{aligned}
\|w\|_{L^q_{t\in I_j}L^r_x} &\le \|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_{t\in I_j}L^r_x} \\
&\quad +C||(-|\tilde{u}+w|^{p-1}(\tilde{u}+w)+|\tilde{u}|^{p-1}\tilde{u}||_{L^{\frac{q}{p}}_{t\in I_j}L^{\frac{r}{p}}_x}+\|e\|_{L^{q_1}_tL^{r_1}_x} \\
&\le \|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_{t\in I_j}L^r_x} \\
&\quad +C||(|\tilde{u}|^{p-1}+|w|^{p-1})w||_{L^{\frac{q}{p}}_{t\in I_j}L^{\frac{r}{p}}_x}+\|e\|_{L^{q_1}_tL^{r_1}_x} \\
&\le \|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_{t\in I_j}L^r_x} \\
&\quad +C||\tilde{u}||^{p-1}_{L^q_{t\in I_j}L^r_x}||w||_{L^q_{t\in I_j}L^r_x}+C||w||^p_{L^q_{t\in I_j}L^r_x}+\|e\|_{L^{q_1}_tL^{r_1}_x}, \\
&\le \|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_{t\in I_j}L^r_x} \\
&\quad +C\delta^{p-1}\|w\|_{L^q_{t\in I_j}L^r_x}+C\|w\|^p_{L^q_{t\in I_j}L^r_x}+C\epsilon.
\end{aligned}$$

If

$$\delta \le (\frac{1}{4C})^{\frac{1}{p-1}},\ (\|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_{t\in I_j}L^r_x}+C\epsilon_0)\le \frac{1}{2}(\frac{1}{4C})^{\frac{1}{p-1}}, \tag{6.7}$$

then

$$\|w\|_{L^q_{t\in I_j}L^r_x}\le 2\|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_{t\in I_j}L^r_x}+2C\epsilon.$$

Now we take $t=t_{j+1}$ in (6.6), and apply $e^{i(t-t_j)\Delta}$ to the both sides , we obtain

$$e^{i(t-t_{j+1})\Delta}w(t_{j+1})=e^{i(t-t_j)\Delta}w(t_j)+i\int_{t_j}^{t_{j+1}}e^{i(t-s)\Delta}W(\cdot,s)ds.$$

Again, with the same method as above, we obtain

$$\|e^{i(t-t_{j+1})\Delta}w(t_{j+1})\|_{L^q_{t\in I_j}L^r_x}\le 2\|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_{t\in I_j}L^r_x}+2C\epsilon.$$

Iterating the above procedure from $j=0$, we have

$$\|e^{i(t-t_j)\Delta}w(t_j)\|_{L^q_tL^r_x}\le 2^j\|e^{i(t-t_0)\Delta}w(t_0)\|_{L^q_tL^r_x}+(2^j-1)2C\epsilon\le 2^{j+2}C\epsilon.$$

To accommodate the second part of (6.7) for all intervals $I_j,\ 0\le j\le N-1$, we require that

$$2^{N+2}C\epsilon_0\le (\frac{1}{4C})^{\frac{1}{p-1}}, \tag{6.8}$$

and we obtain the result easily.

Now we recall the parameter dependence of parameters: We choose δ to meet the first part of (6.7). Given A, the number of the interval N is determined, and the inequality (6.8) tells how small ϵ_0 should be taken in terms of $N(A)$.

□

Note from the proof above that, the parameters in Lemma 6.6 is not dependent on T . As is stated in [8] for $N = 3$, besides the H^1 asymptotic orthogonality (6.2) at $t = 0$, this property can be extended to the NLS flow for $0 \leq t \leq T$ as an application of Lemma 6.6 with an constant $A = A(T)$ dependent on T (but only through A). As for the general Mass-supercritical and Energy-subcritical case, we can prove the following similar result:

Lemma 6.7. *(H^1 Pythagorean Decomposition Along the NLS Flow). Suppose $\phi_n(x)$ be a uniformly bounded sequence in H^1 . Fix any time $0 < T < \infty$. Suppose that $u_n(t) \equiv NLS(t)\phi_n$ exists up to time T for all n and*

$$\lim_{n\to\infty} \|\nabla u_n(t)\|_{L^\infty([0,T];L^2)} < \infty.$$

Let $W_n^M(t) \equiv NLS(t)W_n^M$. Then, for all j , $v^j(t) \equiv NLS(t)\psi^j$ exist up to time T and for all $t \in [0,T]$,

$$\|\nabla u_n\|_2^2 = \sum_{j=1}^{M} \|\nabla v^j(t - t_n^j)\|_2^2 + \|\nabla W_n^M(t)\|_2^2 + o_n(1). \tag{6.9}$$

Here, $o_n(1) \to 0$ uniformly on $0 \leq t \leq T$.

Proof. Let M_0 be such that for $M_1 \geq M_0$ and for δ_{sd} in Lemma 2.3, we have

$$\|NLS(t)W_n^{M_1}\|_{S(\dot H^{s_c})} \leq \delta_{sd}/2$$

and $\|v^j\|_{S(\dot H^{s_c})} \leq \delta_{sd}$ for $j > M_0$. Reorder the first M_0 profiles and introduce an index M_2 , $0 \leq M_2 \leq M_0$, such that

- For each $0 \leq j \leq M_2$ we have $t_n^j = 0$.(There is no j in this category if $M_2 = 0$.)
- For each $M_2+1 \leq j \leq M_0$ we have $|t_n^j| \to \infty$. (There is no j in this category if $M_2 = M_0$.)

By definition of M_0, $v^j(t)$ for $j > M_0$ scatters in both time directions. We claim that for fixed T and $M_2 + 1 \leq j \leq M_0$, $\|v^j(t - t_n^j)\|_{S(\dot H^{s_c};[0,T])} \to 0$ as $n \to \infty$. Indeed, take the case $t_n^j \to +\infty$ for example. By Proposition 6.5, $\|v^j(-t)\|_{S(\dot H^{s_c};[0,\infty))} < \infty$. Then for $q < \infty$, $\|v^j(-t)\|_{L^q([0,\infty);L^r)} < \infty$ implies $\|v^j(t-t_n^j)\|_{L^q([0,T];L^r)} \to 0$. On the other hand, since $v^j(t)$ in Proposition 6.5 is constructed by the existence of wave operators which converge in H^1 to a linear flow at $-\infty$, then the $L^{\frac{2N}{N-2s_c}}$ decay of the linear flow implies immediately that $\|v^j(t - t_n^j)\|_{L^\infty([0,T];L^{\frac{2N}{N-2s_c}})} \to 0$.

Let $B = \max(1, \lim_n \|\nabla u_n\|_{L^\infty([0,T];L^2)})$. For each $1 \leq j \leq M_2$, define $T^j \leq T$ to be the maximal forward time on which $\|\nabla v^j\|_{L^\infty([0,T^j];L^2)} \leq 2B$. Let $\widetilde T = \min_{1\leq j\leq M_2} T^j$, and if $M_2 = 0$, we just take $\widetilde T = T$. Note that if we have proved (6.9) holds for $T = \widetilde T$, then by definition of T^j , using the continuity arguments, it follows from (6.9) that for each $1 \leq j \leq M_2$, we have $T^j = T$. Hence $\widetilde T = T$. Thus, for the remainder of the proof, we just work on $[0, \widetilde T]$.

For each $1 \leq j \leq M_2$, $\|v^j\|_{L^\infty([0,\widetilde T];L^2)} = \|\psi^j\|_2 \leq \lim_n \|\phi_n\|_2$ by (6.2) . Now, in view of the notation of $S(\dot H^{s_c}; [0, \widetilde T])$ and Remark 6.1, we will give the $S(\dot H^{s_c}; [0, \widetilde T])$-norm boundedness of v^j in two cases:

Let $(\tilde{q},\tilde{r})=(\frac{(p-1)(N+2)}{2},\frac{(p-1)(N+2)}{2})$. Case 1, if $\tilde{r}\geq\frac{2N}{N-2}$ and thus $(\frac{2}{1-s_c},\frac{2N}{N-2})\in\Lambda_{s_c}$, then

$$\begin{aligned}\|v^j(t)\|_{S(\dot{H}^{s_c};[0,\widetilde{T}])}&\leq C(\|v^j\|_{L^\infty([0,\widetilde{T}];L^{\frac{2N}{N-2s_c}})}+\|v^j\|_{L^{\frac{2}{1-s_c}}([0,\widetilde{T}];L^{\frac{2N}{N-2}})})\\&\leq C(\|v^j\|^{1-s_c}_{L^\infty([0,\widetilde{T}];L^2)}\|\nabla v^j\|^{s_c}_{L^\infty([0,\widetilde{T}];L^2)}+\widetilde{T}^{\frac{1-s_c}{2}}\|\nabla v^j\|_{L^\infty([0,\widetilde{T}];L^2)})\\&\leq C(1+\widetilde{T}^{\frac{1-s_c}{2}})B.\end{aligned}$$

Case 2, if on the other hand $\tilde{r}<\frac{2N}{N-2}$. Since clearly $(\tilde{q},\tilde{r})\in\Lambda_{s_c}$,

$$\begin{aligned}\|v^j(t)\|_{S(\dot{H}^{s_c};[0,\widetilde{T}])}&\leq C(\|v^j\|_{L^\infty([0,\widetilde{T}];L^{\frac{2N}{N-2s_c}})}+\|v^j\|_{L^{\frac{(p-1)(N+2)}{2}}([0,\widetilde{T}];L^{\frac{(p-1)(N+2)}{2}})})\\&\leq C(\|v^j\|^{1-s_c}_{L^\infty([0,\widetilde{T}];L^2)}\|\nabla v^j\|^{s_c}_{L^\infty([0,\widetilde{T}];L^2)}+\widetilde{T}^{\frac{(p-1)(N+2)}{2}}\|\nabla v^j\|_{L^\infty([0,\widetilde{T}];L^2)})\\&\leq C(1+\widetilde{T}^{\frac{(p-1)(N+2)}{2}})B.\end{aligned}$$

For fixed M , let

$$\tilde{u}_n(x,t)=\sum_{j=1}^{M}v^j(x-x_n^j,t-t_n^j),$$

and let

$$e_n=i\partial_t\tilde{u}_n+\Delta\tilde{u}_n+|\tilde{u}_n|^{p-1}\tilde{u}_n.$$

We claim that there exists $A=A(\widetilde{T})$ (independent of M)such that for all $M>M_0$, there exists $n_0=n_0(M)$ such that for all $n>n_0$,

$$\|\tilde{u}_n\|_{S(\dot{H}^{s_c};[0,\widetilde{T}])}\leq A.$$[1]

Furthermore, we also claim that for each $M>M_0$ and $\epsilon>0$, there exists $n_1=n_1(M,\epsilon)$ such that for $n>n_1$ and for some $(q,r)\dot{H}^{-s_c}$ admissible,

$$\|e_n\|_{L^{q'}([0,\widetilde{T}];L^{r'})}\leq\epsilon.$$

Both of the two claims have exactly been verified in [18](in the proof of Proposition 4.4 there) , we shall not prove them here again. Moreover, since $u_n(0)-\tilde{u}_n(0)=W_n^M$, there exists $M'=M'(\epsilon)$ large enough such that for each $M>M'$ there exists $n_2=n_2(M')$ such that for $n>n_2$,

$$\|e^{it\Delta}(u(0)-\tilde{u}(0))\|_{S(\dot{H}^{s_c};[0,\widetilde{T}])}\leq\epsilon.$$

For $A=A(\widetilde{T})$ in the first claim, Lemma 6.6 gives us $\epsilon_0=\epsilon_0(A)\ll 1$. We select an arbitrary $\epsilon\leq\epsilon_0$ and obtain from above arguments an index $M'=M'(\epsilon)$. Now select an arbitrary $M>M'$, and set $n'=\max(n_0,n_1,n_2)$. Then by Lemma 6.6 and the above arguments, for $n>n'$, we have

$$\|u_n-\tilde{u_n}\|_{S(\dot{H}^{s_c};[0,T])}\leq c(\widetilde{T})\epsilon. \tag{6.10}$$

[1]We in fact prove both $\|\tilde{u}_n\|_{L^{\frac{(p-1)(N+2)}{2}}([0,\widetilde{T}];L^{\frac{(p-1)(N+2)}{2}})}$ and $\|\tilde{u}_n\|_{L^\infty([0,\widetilde{T}];L^{\frac{(2N)(N-2s_c)}{2}})}$ are bounded, and thus, by interpolation, for any $(q,r)\in\Lambda_{s_c}$ $(q\geq r)$, we obtain the $\|\tilde{u}_n\|_{L^q([0,\widetilde{T}];L^r)}$ bound.

In order to obtain the $\|\nabla\tilde{u}_n\|_{L^\infty([0,\widetilde{T}];L^2)}$ bound, we also have to discuss $j \geq M_2+1$. As is noted in the first paragraph of the proof, $\|v^j(t-t_n^j)\|_{S(\dot{H}^{s_c};[0,\widetilde{T}])} \to 0$ as $n \to \infty$. By Strichartz estimate we can easily get $\|\nabla v^j(t-t_n^j)\|_{L^\infty([0,\widetilde{T}];L^2)} \leq C\|\nabla v^j(-t_n^j)\|_2$. By the pairwise divergence of parameters,

$$\begin{aligned}\|\nabla\tilde{u}_n\|^2_{L^\infty([0,\widetilde{T}];L^2)} &= \sum_{j=1}^{M_2}\|\nabla v^j(t)\|^2_{L^\infty([0,\widetilde{T}];L^2)} + \sum_{M_2+1}^{M}\|\nabla v^j(t-t_n^j)\|^2_{L^\infty([0,\widetilde{T}];L^2)} + o_n(1)\\ &\leq C\left(M_2B^2 + \sum_{M_2+1}^{M}\|\nabla NLS(-t_n^j)\psi^j\|_2^2 + o_n(1)\right)\\ &\leq C\left(M_2B^2 + \|\nabla\phi_n\|_2^2 + o_n(1)\right)\\ &\leq C\left(M_2B^2 + B^2 + o_n(1)\right).\end{aligned}$$

Note that $\frac{2N}{N-2s_c} < p+1 < \frac{2N}{N-2}$, then for some $0<\theta<1$ and from (6.10) we have

$$\begin{aligned}\|u_n-\tilde{u}_n\|_{L^\infty([0,\widetilde{T}];L^{p+1})} &\leq C\left(\|u_n-\tilde{u}_n\|^\theta_{L^\infty([0,\widetilde{T}];L^{\frac{2N}{N-2s_c}})}\|\nabla(u_n-\tilde{u}_n)\|^{1-\theta}_{L^\infty([0,\widetilde{T}];L^2)}\right) \quad (6.11)\\ &\leq c(\widetilde{T})^\theta\left(M_2B^2+B^2+o_n(1)\right)^{\frac{1-\theta}{2}}\epsilon^\theta.\end{aligned}$$

Now in the sequel we first replace the large parameter M in the notation $\tilde{u}_n$ and all other arguments above for M_1 which appears at the beginning of our proof. Then for any fixed M, we will prove (6.9) on $[0,\widetilde{T}]$. In fact, we need only to establish that, for each $t\in[0,\widetilde{T}]$,

$$\|u_n\|^{p+1}_{p+1} = \sum_{j=1}^{M}\|v^j(t-t_n^j)\|^{p+1}_{p+1} + \|W_n^M(t)\|^{p+1}_{p+1} + o_n(1). \quad (6.12)$$

Since then by (6.3) and the energy conservation we have

$$E(u_n(t)) = \sum_{j=1}^{M}E(v^j(t-t_n^j)) + E(W_n^M(t)) + o_n(1). \quad (6.13)$$

Thus (6.12) combined with (6.13) gives (6.9) , which completes our proof. So now what is the remainder is to establish (6.12).

We first apply the perturbation theory Lemma 6.6 to $u_n = W_n^M$ and $\tilde{u}_n = \sum_{j=M+1}^{M_1} v^j(t-t_n^j)$. For any fixed $M<M_1$, by the profile composition (6.1) and the definition of $W_n^M(t)$ and $v^j(t)$, similar to the above two claims and the arguments followed, we obtain

$$\|W_n^M(t) - \sum_{j=M+1}^{M_1} v^j(t-t_n^j)\|_{p+1} \to 0 \quad as \quad n\to\infty.$$

Then by the pairwise divergence of parameters,

$$\begin{aligned}\|u_n\|_{p+1}^{p+1} &= \|\tilde{u}_n\|_{p+1}^{p+1} + o_n(1)\\ &= \|\sum_{j=1}^{M_1} v^j(t-t_n^j)\|_{p+1}^{p+1} + o_n(1)\\ &= \sum_{j=1}^{M} \|v^j(t-t_n^j)\|_{p+1}^{p+1} + \|\sum_{j=M+1}^{M_1} v^j(t-t_n^j)\|_{p+1}^{p+1} + o_n(1)\\ &= \sum_{j=1}^{M} \|v^j(t-t_n^j)\|_{p+1}^{p+1} + \|W_n^M(t)\|_{p+1}^{p+1} + o_n(1).\end{aligned}$$

If on the other hand $M \geq M_1$, we then easily get from the selection of M_1 at the beginning of our proof that $\|W_n^M(t)\|_{p+1} = o_n(1)$ and (6.11) implies (6.12).

□

Lemma 6.8. *(Profile Reordering).Let $\phi_n(x)$ be a bounded sequence in H^1 and let $\lambda_0 > 1$. Suppose that $M(\phi_n) = M(Q)$, $E(\phi_n)/E(Q) = \omega_1\lambda_n^2 - \omega_2\lambda_n^{\frac{N(p-1)}{2}}$ with $\lambda_n \geq \lambda_0 > 1$ and $\|\nabla\phi_n\|_2/\|\nabla Q\|_2 \geq \lambda_n$ for each n. Then, for a given M, the profiles can be reordered so that there exist $1 \leq M_1 \leq M_2 \leq M$ and*
(1) For each $1 \leq j \leq M_1$, we have $t_n^j = 0$ and $v^j(t) \equiv NLS(t)\psi^j$ does not scatter as $t \to +\infty$. (We in fact assert that at least one j belongs to this category.)
(2) For each $M_1 + 1 \leq j \leq M_2$, we have $t_n^j = 0$ and $v^j(t)$ scatters as $t \to +\infty$. (There is no j in this category if $M_2 = M_1$.)
(3) For each $M_2 + 1 \leq j \leq M$ we have $|t_n^j| \to \infty$. (There is no j in this category if $M_2 = M$.)

Proof. Firstly, we prove that there exists at least one j such that t_n^j converges as $n \to \infty$. In fact,

$$\begin{aligned}\frac{\|\phi_n\|_{p+1}^{p+1}}{\|Q\|_{p+1}^{p+1}} &= -\frac{1}{\omega_2}\frac{E(\phi_n)}{E(Q)} + \frac{N(p-1)}{4}\frac{\|\nabla\phi_n\|_2^2}{\|\nabla Q\|_2^2}\\ &\geq -\frac{1}{\omega_2}\left(\omega_1\lambda_n^2 - \omega_2\lambda_n^{\frac{N(p-1)}{2}}\right) + \frac{N(p-1)}{4}\lambda_n^2\\ &= \lambda_n^{\frac{N(p-1)}{2}} \geq \lambda_0^{\frac{N(p-1)}{2}} > 1.\end{aligned} \tag{6.14}$$

If $|t_n^j| \to \infty$, then $\|NLS(-t_n^j)\psi^j\|_{p+1} \to 0$ and (6.4) implies our claim. Now if j is such that t_n^j converges as $n \to \infty$, then we might as well assume $t_n^j = 0$.

Reorder the profiles ψ^j so that for $1 \leq j \leq M_2$, we have $t_n^j = 0$, and for $M_2 + 1 \leq j \leq M$ we have $|t_n^j| \to \infty$. What is the remainder is to show that there exists one j, $1 \leq j \leq M_2$, such that $v^j(t)$ does not scatter as $t \to +\infty$. To the contrary, if for all $1 \leq j \leq M_2$, $v^j(t)$ scatters, then we have $\lim_{t\to+\infty} \|v^j(t)\|_{p+1} = 0$. Let t_0 be sufficiently large so that for

all $1 \le j \le M_2$, we have $\|v^j(t_0)\|_{p+1}^{p+1} \le \epsilon/M_2$. The L^{p+1} orthogonality (6.12) along the NLS flow and an argument as (6.14) imply

$$\lambda_0^{\frac{N(p-1)}{2}} \|Q\|_{p+1}^{p+1} \le \|u_n(t_0)\|_{p+1}^{p+1}$$
$$= \sum_{j=1}^{M_2} \|v^j(t_0)\|_{p+1}^{p+1} + \sum_{j=M_2+1}^{M} \|v^j(t_0 - t_n^j)\|_{p+1}^{p+1} + \|W_n^M(t_0)\|_{p+1}^{p+1} + o_n(1).$$

We know from Proposition 6.5 that,as $n \to +\infty$, $\sum_{j=M_2+1}^{M} \|v^j(t_0 - t_n^j)\|_{p+1}^{p+1} \to 0$, and thus we have

$$\lambda_0^{\frac{N(p-1)}{2}} \|Q\|_{p+1}^{p+1} \le \epsilon + \|W_n^M(t_0)\|_{p+1}^{p+1} + o_n(1).$$

This gives a contradiction since $W_n^M(t)$ is a scattering solution.

□

7. Inductive Argument and Existence of a Critical Solution

We now begin to prove Theorem 1.1. Note from Remark 1.2 that we have reduced Theorem 1.1 to the case $P(u) = 0$, thus we first give some definitions :

Definition 7.1. Let $\lambda > 1$. We say that $\exists GB(\lambda, \sigma)$ holds if there exists a solution $u(t)$ to (1.1) such that

$$P(u) = 0, \quad M(u) = M(Q), \quad \frac{E(u)}{E(Q)} = \omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}}$$

and

$$\lambda \le \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2} \le \sigma \quad for\ all \quad t \ge 0.$$

$\exists GB(\lambda, \sigma)$ means that there exist solutions with energy $\omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}}$ globally bounded by σ. Thus by Proposition 5.1 , $\exists GB(\lambda, \lambda(1 + \rho_0(\lambda_0)))$ is false for all $\lambda \ge \lambda_0 > 1$.

The statement $\exists GB(\lambda, \sigma)$ is false is equivalent to say that for every solution $u(t)$ to (1.1) with $M(u) = M(Q)$ and $E(u)/E(Q) = \omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}}$ such that $\|\nabla u(t)\|_2/\|\nabla Q\|_2 \ge \lambda$ for all t, there must exists a time $t_0 \ge 0$ such that $\|\nabla u(t_0)\|_2/\|\nabla Q\|_2 \ge \sigma$. By resetting the initial time, we can find a sequence $t_n \to \infty$ such that $\|\nabla u(t_n)\|_2/\|\nabla Q\|_2 \ge \sigma$ for all n.

Note that if $\lambda \le \sigma_1 \le \sigma_2$, then $\exists GB(\lambda, \sigma_2)$ is false implies $\exists GB(\lambda, \sigma_1)$ is false. We will induct on the statement and define a threshold.

Definition 7.2. (The Critical Threshold.) Fix $\lambda_0 > 1$. Let $\sigma_c = \sigma_c(\lambda_0)$ be the supremum of all $\sigma > \lambda_0$ such that $\exists GB(\lambda, \sigma)$ is false for all λ such that $\lambda_0 \le \lambda \le \sigma$.

Proposition 5.1 implies that $\sigma_c(\lambda_0) > \lambda_0$. Let $u(t)$ be any solution to (1.1) with $P(u) = 0$, $M(u) = M(Q)$, $E(u)/E(Q) \le \omega_1 \lambda_0^2 - \omega_2 \lambda_0^{\frac{N(p-1)}{2}}$ and $\|\nabla u(0)\|_2/\|\nabla Q\|_2 > 1$. If $\lambda_0 > 1$ and $\sigma_c = \infty$, we claim that there exists a sequence of times t_n such that $\|\nabla u(t_n)\|_2 \to \infty$. In fact, if not, and let $\lambda \ge \lambda_0$ be such that $E(u)/E(Q) = \omega_1 \lambda^2 - \omega_2 \lambda^{\frac{N(p-1)}{2}}$. Since

there is no sequence t_n such that $\|\nabla u(t_n)\|_2 \to \infty$, there must exists $\sigma < \infty$ such that $\lambda \leq \|\nabla u(t)\|_2/\|\nabla Q\|_2 \leq \sigma$ for all $t \geq 0$, which means that $\exists GB(\lambda, \sigma)$ holds true. Thus $\sigma_c \leq \sigma < \infty$ and we get a contradiction.

In view of the above claim, if we can prove that for every $\lambda_0 > 1$ then $\sigma_c(\lambda_0) = \infty$, we then have in fact proved our Theorem 1.1. Thus, in the sequel, we shall carry it out by contradiction; more precisely , fix $\lambda_0 > 1$ and assume $\sigma_c < \infty$, we shall work toward a absurdity. (It, of course, suffices to do this for λ_0 close to 1, and so we might as well assume that $\lambda_0 < (\frac{\omega_1}{\omega_2})^{\frac{2}{N(p-1)-4}}$, which will be convenient in the sequel.) For that purpose, we need first to obtain the existence of a critical solution:

Lemma 7.3. $\sigma_c(\lambda_0) < \infty$. *Then there exist initial data* $u_{c,0}$ *and* $\lambda_c \in [\lambda_0, \sigma_c(\lambda_0)]$ *such that* $u_c(t) \equiv NLS(t)u_{c,0}$ *is global,* $P(u_c) = 0$, $M(u_c) = M(Q)$, $E(u_c)/E(Q) = \omega_1\lambda_c^2 - \omega_2\lambda_c^{\frac{N(p-1)}{2}}$, *and*

$$\lambda_c \leq \frac{\|\nabla u_c(t)\|_2}{\|\nabla Q\|_2} \leq \sigma_c \quad \textit{for all} \quad t \geq 0.$$

We call u_c a critical solution since by definition of σ_c we have that for all $\sigma < \sigma_c$ and all $\lambda_0 \leq \lambda \leq \sigma$, $\exists GB(\lambda, \sigma)$ is false, i.e., there are no solutions $u(t)$ for which

$$P(u) = 0, \qquad M(u) = M(Q), \qquad \frac{E(u)}{E(Q)} = \omega_1\lambda^2 - \omega_2\lambda^{\frac{N(p-1)}{2}}$$

and

$$\lambda \leq \frac{\|\nabla u(t)\|_2}{\|\nabla Q\|_2} \leq \sigma \quad \textit{for all} \quad t \geq 0.$$

Proof. By definition of σ_c, there exist sequence λ_n and σ_n such that $\lambda_0 \leq \lambda_n \leq \sigma_n$ and $\sigma_n \downarrow \sigma_c$ for which $\exists GB(\lambda_n, \sigma_n)$ holds. This means that there exists $u_{n,0}$ such that $u_n(t) \equiv NLS(t)u_{n,0}$ is global with $P(u_n) = 0$, $M(u_n) = M(Q)$, $E(u_n)/E(Q) = \omega_1\lambda_n^2 - \omega_2\lambda_n^{\frac{N(p-1)}{2}}$, and

$$\lambda_n \leq \frac{\|\nabla u_n(t)\|_2}{\|\nabla Q\|_2} \leq \sigma_n \quad \textit{for all} \quad t \geq 0.$$

The boundedness of λ_n make us pass to a subsequence such that λ_n converges with a limit $\lambda' \in [\lambda_0, \sigma_c]$.

According to Lemma 6.8 where we take $\phi_n = u_{n,0}$, for $M_1 + 1 \leq j \leq M_2$, $v^j(t) \equiv NLS(t)\psi^j$ scatter as $t \to +\infty$ and combined with Proposition 6.5, for $M_2 + 1 \leq j \leq M$, v^j also scatter in one or the other time direction. Thus by the scattering theory, for $M_1 + 1 \leq j \leq M$, we have $E(v_j) = E(\psi_j) \geq 0$ and then by (6.3)

$$\sum_{j=1}^{M_1} E(\psi^j) \leq E(\phi_n) + o_n(1).$$

Thus there exists at least one $1 \leq j \leq M_1$ with

$$E(\psi^j) \leq \max(\lim_n E(\phi_n), 0),$$

which, without loss of generality, we might as well take $j = 1$. Since, by the profile composition, also $M(\psi^1) \leq \lim_n M(\phi_n) = M(Q)$, we then have

$$\frac{M^{\frac{1-s_c}{s_c}}(\psi^1)E(\psi^1)}{M^{\frac{1-s_c}{s_c}}(Q)E(Q)} \leq \max\left(\lim_n \frac{E(\phi_n)}{E(Q)}, 0\right).$$

Thus, there exist $\tilde{\lambda} \geq \lambda_0$ [2]such that

$$\frac{M^{\frac{1-s_c}{s_c}}(\psi^1)E(\psi^1)}{M^{\frac{1-s_c}{s_c}}(Q)E(Q)} = \omega_1 \tilde{\lambda}^2 - \omega_2 \tilde{\lambda}^{\frac{N(p-1)}{2}}.$$

Note that by Lemma 6.8, v^1 does not scatter, so it follows from Theorem 2.5 that $\|\psi^1\|_2^{\frac{1-s_c}{s_c}} \|\nabla \psi^1\|_2 < \|Q\|_2^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2$ cannot hold. Then by Proposition 2.2, we must have $\|\psi^1\|_2^{\frac{1-s_c}{s_c}} \|\nabla \psi^1\|_2 \geq \tilde{\lambda} \|Q\|_2^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2$.

Now if $\tilde{\lambda} > \sigma_c$ and recall that $t_n^1 = 0$, then for all t we know that

$$\tilde{\lambda}^2 \leq \frac{\|v^1(t)\|_2^{2\frac{1-s_c}{s_c}} \|\nabla v^1(t)\|_2^2}{\|Q\|_2^{2\frac{1-s_c}{s_c}} \|\nabla Q\|_2^2} \leq \frac{\|\nabla v^1(t)\|_2^2}{\|\nabla Q\|_2^2} \leq \frac{\sum_{j=1}^M \|\nabla v^j(t - t_n^j)\|_2^2 + \|\nabla W_n^M(t)\|_2^2}{\|\nabla Q\|_2^2}. \quad (7.1)$$

Taking $t = 0$ for example, Lemma 6.7 implies that

$$\tilde{\lambda}^2 \leq \frac{\sum_{j=1}^M \|\nabla v^j(-t_n^j)\|_2^2 + \|\nabla W_n^M\|_2^2}{\|\nabla Q\|_2^2} \leq \frac{\|\nabla u_n(0)\|_2^2}{\|\nabla Q\|_2^2} + o_n(1) \leq \sigma_c^2 + o_n(1)$$

which contradicts the assumption $\tilde{\lambda} > \sigma_c$. Hence we must have $\tilde{\lambda} \leq \sigma_c$.

Now if $\tilde{\lambda} < \sigma_c$, we know from the definition of σ_c that $\exists GB(\tilde{\lambda}, \sigma_c - \delta)$ is false for any $\delta > 0$ sufficiently small, and then there exists a nondecreasing sequence t_k of times such that

$$\lim_k \frac{\|v^1(t_k)\|_2^{\frac{1-s_c}{s_c}} \|\nabla v^1(t_k)\|_2}{\|Q\|_2^{\frac{1-s_c}{s_c}} \|\nabla Q\|_2} \geq \sigma_c.$$

Note that $t_n^1 = 0$, then

$$\begin{aligned} \sigma_c^2 - o_k(1) &\leq \frac{\|v^1(t_k)\|_2^{2\frac{1-s_c}{s_c}} \|\nabla v^1(t_k)\|_2^2}{\|Q\|_2^{2\frac{1-s_c}{s_c}} \|\nabla Q\|_2^2} \leq \frac{\|\nabla v^1(t_k)\|_2^2}{\|\nabla Q\|_2^2} \qquad (7.2) \\ &\leq \frac{\sum_{j=1}^M \|\nabla v^j(t_k - t_n^j)\|_2^2 + \|\nabla W_n^M(t_k)\|_2^2}{\|\nabla Q\|_2^2} \\ &\leq \frac{\|\nabla u_n(t)\|_2^2}{\|\nabla Q\|_2^2} + o_n(1) \\ &\leq \sigma_c^2 + o_n(1), \end{aligned}$$

[2]If $\lim_n E(\phi_n) \geq 0$, we have $\tilde{\lambda} \geq \lambda' \geq \lambda_0$; while in the case $\lim_n E(\phi_n) < 0$, we will have $\tilde{\lambda} \geq (\frac{\omega_1}{\omega_2})^{\frac{2}{N(p-1)-4}} > \lambda_0$ though we might not have $\tilde{\lambda} \geq \lambda'$.

where by Lemma 6.7 we take $n = n(k)$ large. Sending $k \to \infty$ and hence $n(k) \to \infty$, we conclude that all inequalities must be equalities. Thus we conclude that $W_n^M(t_k) \to 0$ in H^1, $M(v^1) = M(Q)$ and $v^j \equiv 0$ for all $j \geq 2$. Thus easily $P(v^1) = P(u_n) = 0$. On the other hand if $\tilde{\lambda} = \sigma_c$, we need not the inductive hypothesis but, similar to (7.1), obtain

$$\sigma_c^2 \leq \frac{\sum_{j=1}^M \|\nabla v^j(-t_n^j)\|_2^2 + \|\nabla W_n^M\|_2^2}{\|\nabla Q\|_2^2} \leq \frac{\|\nabla u_n(0)\|_2^2}{\|\nabla Q\|_2^2} + o_n(1) \leq \sigma_c^2 + o_n(1),$$

and then again, we conclude that $W_n^M \to 0$ in H^1, $M(v^1) = M(Q)$ and $v^j \equiv 0$ for all $j \geq 2$. Moreover, by Lemma 6.7, for all t

$$\frac{\|\nabla v^1(t)\|_2^2}{\|\nabla Q\|_2^2} \leq \lim_n \frac{\|\nabla u_n(t)\|_2^2}{\|\nabla Q\|_2^2} \leq \sigma_c^2.$$

Hence, we take $u_{c,0} = v^1(0) = \psi^1$ and $\lambda_c = \tilde{\lambda}$ to complete our proof.

□

8. Concentration of Critical Solutions and Proof of Theorem 1.1

In this section, we will finally prove Theorem 1.1 by virtue of the precompactness of the flow of the critical solution. To simplify notation, we take $u(t) = u_c(t)$ in the sequel.

Lemma 8.1. *There exists a path $x(t)$ in $\mathbb{R}^N$ such that*

$$K \equiv \{u(t, \cdot - x(t)) | t \geq 0\} \subset H^1$$

is precompact in H^1.

Proof. As is showed in [3] , it suffices to prove that for each sequence of times $t_n \to \infty$, there exists a sequence x_n such that, by passing to a subsequence, $u(t_n, \cdot - x_n)$ converges in H^1.

Taking $\phi_n = u(t_n)$ in Lemma 6.8 and by definition of $u(t) = u_c(t)$, similar to the proof of Lemma 7.3, we obtain that there exists at least one $1 \leq j \leq M_1$ with

$$E(\psi^j) \leq \max(\lim_n E(\phi_n), 0).$$

Without loss of generality, we can take $j = 1$. Since, also $M(\psi^1) \leq \lim_n M(\phi_n) = M(Q)$, there exist $\tilde{\lambda} \geq \lambda_0$ such that

$$\frac{M^{\frac{1-s_c}{s_c}}(\psi^1)E(\psi^1)}{M^{\frac{1-s_c}{s_c}}(Q)E(Q)} = \omega_1 \tilde{\lambda}^2 - \omega_2 \tilde{\lambda}^{\frac{N(p-1)}{2}}.$$

Note that by Lemma 6.8, v^1 does not scatter, so we must have $\|\psi^1\|_2 \|\nabla \psi^1\|_2 \geq \tilde{\lambda} \|Q\|_2 \|\nabla Q\|_2$. Then by the same way as in the proof of Lemma 7.3 , we get that $W_n^M(t_k) \to 0$ in H^1 and $v^j \equiv 0$ for all $j \geq 2$. Since we know that $W_n^M(t)$ is a scattering solution , this implies that

$$W_n^M(0) = W_n^M \to 0 \quad in \quad H^1. \tag{8.1}$$

Consequently, we have

$$u(t_n) = NLS(-t_n^1)\psi^1(x - x_n^1) + W_n^M(x).$$

Note that by Lemma 6.8, $t_n^1 = 0$, and thus

$$u(t_n, x + x_n^1) = \psi^1(x) + W_n^M(x + x_n^1).$$

This equality and (8.1) imply our conclusion. □

Using the uniform-in-time H^1 concentration of $u(t) = u_c(t)$ and by changing of variables, we can easily get

Corollary 8.2. *For each $\epsilon > 0$, there exists $R > 0$ such that for all t,*

$$\|u(t, \cdot - x(t))\|_{H^1(|x| \geq R)} \leq \epsilon.$$

With the localization property of u_c, we show, similar to [8], that u_c must blow up in finite time using the same method as that in the proof of Proposition 3.2, which contradicts the boundedness of u_c in H^1. Hence, u_c cannot exist and $\sigma_c = \infty$. As is argued in section 7, this indeed completes the proof of Theorem 1.1.

Appendix A. Nonzero Momentum

Suppose that the solution $u(x,t)$ with $M(u) = M(Q)$, $P(u) \neq 0$. Applying Galilean transform to $u(x,t)$, we obtain a new solution $\tilde{u}(x,t)$:

$$\tilde{u}(x,t) = e^{ix\cdot\xi_0} e^{-it|\xi_0|^2} u(x - 2\xi_0 t, t).$$

Take $\xi_0 = -\frac{P(u)}{M(u)}$ and we get

$$P(\tilde{u}) = 0, \quad M(\tilde{u}) = M(u) = M(Q), \quad \|\nabla\tilde{u}\|_2^2 = \|\nabla u\|_2^2 - \frac{P(u)^2}{M(u)}$$

and

$$E(\tilde{u}) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{M(u)}{2}\left(\xi_0 + \frac{P(u)}{M(u)}\right)^2 - \frac{P(u)^2}{2M(u)} = E(u) - \frac{1}{2}\frac{P(u)^2}{M(u)}.$$

Thus this choice of ξ_0 make $E(\tilde{u})$ attain its lowest value under any choice of $\xi_0 \in \mathbb{R}^N$. And as is stated in [8], $E(\tilde{u}) < E(u) < E(Q)$ implies that we should always implement this transformation to maximize the applicability of Proposition 2.2.

Now what we should do is to show that if the dichotomy of Proposition 2.2 was already valid for u, then the selection of case (1) versus (2) in Proposition 2.2 is preserved under the Galilean transformation.

Suppose $M(u) = M(Q)$, $E(u) < E(Q)$ and $P(u) \neq 0$. Define $\tilde{u}(x,t)$ as above. Let λ_-, λ be defined in terms of $E(u)$ by (2.6) and $\eta(t)$ in terms of $u(t)$ by (2.4). Let$\tilde{\lambda}_-, \tilde{\lambda}$ and $\tilde{\eta}(t)$ be the same quantities associated to $\tilde{u}$.

Firstly, suppose that case (1) of Proposition 2.2 holds for u, which in particular implies that $\eta(t) < 1$ for all t. But clearly $\tilde{\eta}(t) < \eta(t) < 1$, thus, case (1) of Proposition 2.2 holds for $\tilde{u}$ also.

Now conversely, suppose that case (1) of Proposition 2.2 holds for $\tilde{u}$, then $\tilde{\eta}(t)^2 \le \tilde{\lambda}_-^2$ for all t. We claim that

$$\eta(t)^2 = \frac{\|\nabla u\|_2^2}{\|\nabla Q\|_2^2} = \tilde{\eta}(t)^2 + \frac{P(u)^2}{2M(u)\|\nabla Q\|_2^2} = \tilde{\eta}(t)^2 + \frac{P(u)^2}{2\omega_1 M(u)E(Q)} \le \lambda_-^2.$$

Indeed, this reduced to an algebraic problem now. Denote $\alpha = \frac{E(u)}{E(Q)}$ and $\beta = \frac{P(u)^2}{M(u)E(Q)}$. Then $\tilde{\lambda}_-$ is the smaller root of the equation:

$$\omega_1 \tilde{\lambda}_-^2 - \omega_2 \tilde{\lambda}_-^{\frac{N(p-1)}{2}} = \frac{E(\tilde{u})}{E(Q)} = \frac{E(u)}{E(Q)} - \frac{P(u)^2}{2M(u)E(Q)} = \alpha - \frac{\beta}{2},$$

while λ_- is the smaller root of

$$\omega_1 \lambda_-^2 - \omega_2 \lambda_-^{\frac{N(p-1)}{2}} = \frac{E(u)}{E(Q)} = \alpha.$$

Let the function $f(x) = \omega_1 x - \omega_2 x^{\frac{N(p-1)}{4}}$. Observe that the above claim follows if we could prove that $f(\tilde{\lambda}_-^2 + \frac{\beta}{2\omega_1}) \le f(\lambda_-^2)$. Equivalently, it suffices to show $f(\tilde{\lambda}_-^2 + \frac{\beta}{2\omega_1}) \le f(\tilde{\lambda}_-^2) + \frac{\beta}{2}$, or

$$f(\tilde{\lambda}_-^2 + \frac{\beta}{2\omega_1}) - f(\tilde{\lambda}_-^2) \le \frac{\beta}{2}. \tag{A.1}$$

The left hand side of (A.1) is $\frac{\beta}{2} - \omega_2 \left((\tilde{\lambda}_-^2 + \frac{\beta}{2\omega_1})^{\frac{N(p-1)}{4}} - (\tilde{\lambda}_-^2)^{\frac{N(p-1)}{4}} \right)$ which is certainly no larger than $\frac{\beta}{2}$ since $p - 1 > \frac{4}{N}$, and we conclude our claim.

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